Travelling heteroclinic waves in a Frenkel-Kontorova chain with anharmonic on-site potential

Boris Buffoni, Hartmut Schwetlick and Johannes Zimmer*

Abstract

The Frenkel-Kontorova model for dislocation dynamics from 1938 is given by a chain of atoms, where neighbouring atoms interact through a linear spring and are exposed to a smooth periodic on-site potential. A dislocation moving with constant speed corresponds to a heteroclinic travelling wave, making a transition from one well of the on-site potential to another. The ensuing system is nonlocal, nonlinear and nonconvex. We present an existence result for a class of smooth nonconvex on-site potentials. Previous results in mathematics and mechanics have been limited to on-site potentials with harmonic wells. To overcome this restriction, we propose a novel approach: we first develop a global centre manifold theory for anharmonic wave trains, then parametrise the centre manifold to obtain asymptotically correct approximations to the solution sought, and finally obtain the heteroclinic wave via a fixed point argument.
Mathematics Subject Classification: 37K60, 34C37, 35B20, 58F03, 70H05

1 Introduction

In 1938, Frenkel and Kontorova [9] proposed a model for plastic deformations and twinning, given by an infinite chain of nonlinear oscillators linearly coupled to their nearest neighbours,

$$\ddot{\upsilon}_j(t) = \gamma\left[(\upsilon_{j+1}(t)-\upsilon_j(t))-(\upsilon_j(t)-\upsilon_{j-1}(t))\right]-g'(\upsilon_j(t)). \tag{1}$$

These are Newton's equation of motion for atom $j \in \mathbb{Z}$ with mass 1; γ is the elastic modulus of the elastics springs and g is smooth and periodic.

Travelling waves as particularly simple forms of coherent motion; here they are of the form $\upsilon_j(t) = u(j-ct)$ with some travelling wave profile u. Equation (1) written in travelling wave coordinates $x := j-ct$, with c being the wave speed, is

$$c^2u''(x)-\gamma\Delta_D u(x)+g'(u(x)) = 0, \tag{2}$$

*University of Bath, Department of Mathematical Sciences, Bath BA2 7AY, United Kingdom, zimmer@maths.bath.ac.uk

where Δ_D is the discrete Laplacian

$$(\Delta_D u)(x) := u(x+1) - 2u(x) + u(x-1). \tag{3}$$

In the original paper [9], the force of the on-site potential is (in suitable units) $g'(u) = \sin(2\pi u)$.

Equation (2) is an advance-delay differential-difference equation of Hamiltonian nature, nonlocal and nonlinear. Proving the existence of small solutions to (2) has been a major challenge, accomplished only in 2000 in the seminal paper by Iooss and Kirchgässner [11]. They establish the existence of *small* amplitude solutions, under the convexity assumption $g''(0) > 0$. In particular, Iooss and Kirchgässner prove the existence of nanopterons, that is, localised waves which are superimposed to a periodic ("phonon") wave train. Another remarkable result is the existence of breathers (spatially localised time-periodic solutions) by MacKay and Aubry [16]. There is a wealth of studies of Frenkel-Kontorova models. We refer the reader to the monograph by Braun [2] and only mention more recent results for sliding states by Qin for a forced Frenkel-Kontorova chain, both with and without damping [17, 18] and periodic travelling waves (wave trains) by Fečkan and Rothos [6].

The mathematical theory of existence of travelling wave dislocation as originally posed in [9], however, is still largely open. One reason is that dislocations are large solutions, making the transition from one well of g to another, and therefore experience the nonconvexity of on-site potential. We highlight a few results for the analysis of travelling dislocations for the chain (2). An early study is that of Frank and van der Merwe [8], where the continuum approximation of (2), the sine-Gordon equation, is analysed. Rigorously, the dangers of relying on the PDE counterpart of a lattice equation were realised decades later (though Schrödinger pointed out this difference in his ingenious analysis [20]). In particular, Iooss and Kirchgässner [11] prove the existence of infinitely many types of travelling waves which do not persist in the continuum approximation. Friesecke and Wattis [10] study the Fermi-Pasta-Ulam-Tsingou (FPUT) chain (nonlinear interaction between nearest neighbour atoms and $g \equiv 0$) and obtain the remarkable result that in a spatially discrete setting, solitary waves exist quite generically, not just for integrable systems (such as the so-called Toda lattice). Recently, explicit solitary waves have also been constructed for the FPUT chain with piecewise quadratic potential [27].

The analysis of dislocation solutions to the lattice equation (2) relies in previous work on the assumption that g is piecewise quadratic; then the force g' in (2) is piecewise linear and Fourier methods can be applied. We refer the reader to Atkinson and Cabrera [1] (note that some findings of that paper have been questioned in the literature [5]) and extensive work by Truskinovsky and collaborators, both for the Fermi-Pasta-Ulam-Tsingou chain with piecewise quadratic interaction [26] and the Frenkel-Kontorova model [14]. Kresse and Truskinovsky have also studied the case of an on-site potential with different moduli (second derivatives at the minima) [15]. Slepyan has made a number of important contributions, for example [23, 22]. Flytzanis, Crowley, and Celli [7] apply Fourier techniques to a problem where the potential consists of three parabolas, the middle one being concave.

To the best of our knowledge, the original problem of a dislocations exposed to an anharmonic on-site potential has only been amenable to careful numerical investigation [19]. Obviously, mathematically, the use of Fourier tools as for the results discussed in the previous paragraph is no longer possible. Physically, the introduction of such a nonlinearity changes the nature of the system fundamentally, as modes can now mix. The physical interpretation of the result presented in this paper is that despite this change, solutions exist and, remarkably, can be obtained via a perturbation argument from the degenerate case of piecewise quadratic wells (the degeneracy manifests itself in a twofold way, firstly in the prevention of mode mixing and secondly in a singularity of the force g' at the dislocation line; the existence result presented here holds for the physically realistic case of smooth forces). We develop what seems to be a novel approach to prove existence for systems with small nonlinearities (in our case g being anharmonic but close to a piecewise harmonic potential; the reason for this restriction is that we apply a fixed point argument). We first obtain a detailed understanding of wave trains in the anharmonic (but near harmonic) wells of the on-site potential; this is obtained by a global centre manifold description, much in the spirit of the local analysis of Iooss and Kirchgässner [11]. Unlike them, we do not perform a normal form analysis but instead construct a parametrisation of the centre manifold. From this knowledge, it is possible to construct a one-parameter family w_β, $\beta \in [-1, 1]$, of approximate (asymptotically correct, as $x \to \pm\infty$) heteroclinic solutions of the Frenkel-Kontorova travelling wave equation. This step can be seen as a homotopy method from solutions or approximate solutions to the problem with piecewise quadratic wells (the homotopy parameter being $\varepsilon \geq 0$ in Theorem 2.1, even if in the end we do not rely on continuity with respect to ε, but rather on the smallness of $\varepsilon > 0$). This is an unconventional step in the sense that we do not attempt to find a homotopy between solutions to a family of problems, but only between approximate solutions. In a final step, the heteroclinic travelling wave solution is obtained from the approximate solutions via a topological fixed point argument. A key property is that the family w_β satisfies a transversality condition with respect to β (see (28)). This method, in a much simpler setting in which centre manifold theory is not used, was developed in [3].

On an abstract level, the approach developed here allows for the passage from a linear problem to a moderately nonlinear one. We remark that the analysis of the linear problem ((2) with piecewise quadratic on-site potential) is challenging in its own right, and has been solved mathematically by a de-singularisation of the Fourier image of the solution [21, 13]. While the detailed arguments we give below are admittedly rather technical, the method developed here might also be useful for the numerical computation of solutions to such nonlinear problems; indeed, the centre manifold approach can guide the implementation of a path-following technique, while the fixed point argument can for example translate into a gradient descent method.

We have chosen the original Frenkel-Kontorova equation (1) but remark that some of the references given above study a modified model, with an added force. There are also extensions to higher space dimensions, for example [24]. The methodology of this paper should in principle apply to these problems as well.

The result proved here covers cases of (2) with g' anharmonic, periodic and C^∞. As our argument is perturbative in nature, it is not clear whether the particular choice of a trigonometric potential made in [9] is covered; we have no explicit control over the range of perturbations covered. However, while the choice of a trigonometric function as made by Frenkel and Kontorova [9] is natural, there is no intrinsic reason to prefer such an on-site potential. Here, we make for simplicity the choice $\gamma = 1$ and place two neighbouring minima of g at ± 1.

At the end of Section 2, we give a plan of the paper, summarising the required steps and linking them to the relevant sections. Throughout the paper, C a constant that may change from line to line; C is independent of the solution u and of small enough ε.

2 Setting and main result

We can assume that in travelling wave coordinates, the dislocation line is at the origin $x = 0$; then all atoms in the left half-line are in one well of the on-site potential g and all atoms in the right half-plane are in the neighbouring well on the right. It is no loss of generality to consider an on-site potential g with two wells, rather than a periodic one. Indeed, the solutions we obtain for a two-well potential are also solutions for the same equation with a periodic potential. This is implied by our approach to obtain a special two-well solution as a sum of an associated particular solution and a corrector, both being uniformly bounded. Since upper and lower bounds on the solution are available, the solution will also solve the problem for a periodic potential built by extension from the two-well potential. We thus show the existence of heteroclinic waves for

$$c^2 u(x)'' - \Delta_D u(x) + \alpha u(x) - \alpha \psi'(u(x)) = 0, \tag{4}$$

where ψ' is a perturbation of the sign function. This choice is made since for $\psi'(u) = \mathrm{sgn}(u)$, the on-site potential is $\frac{\alpha}{2} \min\big((u+1)^2, (u-1)^2\big)$, being a primitive of the force $\alpha u - \alpha \mathrm{sgn}(u)$. So in this special case the on-site potential is a double-well potential, mimicking two neighbouring wells of the trigonometric potential proposed by Frenkel and Kontorova. Precise assumptions on ψ are stated in Theorem 2.1 below.

To motivate some assumptions in main theorem, we briefly inspect the linear part of (4),

$$u \to Lu := c^2 u'' - \Delta_D u + \alpha u. \tag{5}$$

In Fourier space, L is written as

$$-c^2 k^2 + 2(1 - \cos k) + \alpha = -c^2 k^2 + 4 \sin^2(k/2) + \alpha =: D(k),$$

where D is the *dispersion function.* Let α be given by

$$\alpha := c^2 \left(\frac{\pi}{2}\right)^2 - 2; \tag{6}$$

this choice was also made in [13]. Then trivially

$$k_0 := \frac{\pi}{2} \tag{7}$$

is one root of D and $-k_0$ is another. Furthermore, for $c = 1$, $D'(k) = -2c^2k + 2\sin k$ vanishes only at $k = 0$. Thus, if c is sufficiently close to 1 (we will only consider the case where additionally $c \leq 1$), then D vanishes exactly at k_0 and $-k_0$. This is the key property of D used in this paper.

The main result of the paper is the following theorem.

Theorem 2.1. *We consider the equation* (4),

$$c^2u'' - \Delta_D u + \alpha u - \alpha\psi'(u) = 0$$

on $\mathbb{R}$*, where* Δ_D *is the discrete Laplacian defined in* (3)*. For small* $\varepsilon \in (0, 1/2)$*, the on-site potential* ψ_ε *is assumed to be an even function* $\psi = \psi_\varepsilon \in C^\infty(\mathbb{R}, \mathbb{R})$ *satisfying the following conditions. Let*

$$|\psi''_\varepsilon(u)| \leq 2\varepsilon^{-1} \textit{ for } |u| < \varepsilon, \tag{8}$$

and, for $|u| \geq \varepsilon$,

$$|\psi'_\varepsilon(u) - \operatorname{sgn}(u)| < C\,\varepsilon, \tag{9}$$

and, again for $|u| \geq \varepsilon$,

$$|\psi''_\varepsilon(u)| < C\,\varepsilon, \quad |\psi'''_\varepsilon(u)| < C\,\varepsilon, \quad |\psi^{(4)}_\varepsilon(u)| < C\,\varepsilon, \quad |\psi^{(5)}_\varepsilon(u)| < C\,\varepsilon \tag{10}$$

(there is no condition on $\psi'''(u)$ *for* $|u| < \varepsilon$*).*

Let k_0 *be given by* (7) *and* α *be given by* (6)*. If* $\varepsilon > 0$ *is small enough, then there exists a range of velocities* $c \leq 1$ *close to* 1 *such that for these velocities, there exists a heteroclinic solution to* (4)*. Here heteroclinic means that the asymptotic state near* $-\infty$ *is in one well of the on-site potential while the state near* $+\infty$ *is in the other.*

The proof of Theorem 2.1 is given in Section 5, using results of Sections 3 and 4. Since the proof is convoluted and technical, we give here an outline. To formulate the sequence of steps, we first introduce some notation. We begin by defining exponentially weighted function spaces as in [11]. For $\nu \in \mathbb{R}$, $m \in \{0, 1, 2, \ldots\}$ and a Banach space X, we denote by $E^\nu_m(X)$ the Banach space of functions $f \in C^m(\mathbb{R}, X)$ such that

$$||f||_{E^\nu_m(X)} := \max_{0 \leq j \leq m} ||e^{-\nu|\cdot|} f^{(j)}||_{L^\infty(\mathbb{R}, X)} < \infty. \tag{11}$$

For $X = \mathbb{R}$, functions which decay exponentially at $\pm\infty$ are contained in the spaces $E^\nu_m(X)$, for some negative $\nu < 0$. We also require analogous function spaces where the mth derivative is not continuous, but only in $L^\infty_{loc}(\mathbb{R})$. So let $F^\nu_m(X)$ the Banach space of functions $f \in W^{m,\infty}(\mathbb{R}, X)$ such that

$$||f||_{F^\nu_m(X)} := \max_{0\le j\le m} ||e^{-\nu|\cdot|} f^{(j)}||_{L^\infty(\mathbb{R},X)} < \infty. \tag{12}$$

If, in the definitions above, the function f is only required to be defined on an open subset $A \subset \mathbb{R}$, we shall write $E^\nu_m(A, X)$ and $F^\nu_m(A, X)$ respectively, where $L^\infty(\mathbb{R}, X)$ is replaced by $L^\infty(A, X)$ in these definitions.

Step 1: Special (degenerate) case, $\varepsilon = 0$. In the limit case $\varepsilon = 0$, ψ is not smooth at 0 by (9), as $\psi'(x) = \mathrm{sgn}(x)$; the choice of $\psi(0)$ is immaterial. Also, ψ satisfies $\psi'(\pm 1) = \pm 1$ and $\psi''(u) = 0$ on $(-\infty, 0)$ and on $(0, \infty)$. We use an existence result [13] for heteroclinic odd solutions $u_p \in H^2_{\mathrm{loc}}(\mathbb{R})$ for the special case $\psi' = \mathrm{sgn}$ in (4),

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \mathrm{sgn}(u) = 0 \tag{13}$$

on $\mathbb{R}$. The parameters α, k_0 and c are as in Theorem 2.1.

Obviously, the core difficulty of (13) is the nonlinearity in the interval $s \in (-1, 1)$. Indeed, for $|\lambda| < 1$ and $\theta \in [0, 2\pi)$, trivially $1 + \lambda \sin(k_0 x + \theta)$ is a solution to (13) on $[1, \infty)$ and $-1 + \lambda \sin(k_0 x - \theta)$ is a solution on $(-\infty, -1]$. From [13], we will use that there exists a function u_p that solves (13), and satisfies

$$\lim_{x\to\pm\infty} (u_p(x) \mp 1 - \lambda \sin(k_0 x \pm \theta)) = 0 \tag{14}$$

for some λ and θ.

The core argument for the next Steps 2–4 is to build a particular family of approximate solutions $w_\beta \in W^{2,\infty}(\mathbb{R})$, which are C^1 as a function of $\beta \in [-1, 1]$, and asymptotically, as $x \to \pm\infty$, they approximate a heteroclinic travelling wave solution. However, they are allowed to be far from a solution near the dislocation, $x = 0$. Step 2 provides such a family for $\varepsilon = 0$, Step 3 extends this existence result for the case $\varepsilon > 0$ we are interested in, and Step 4 uses this family of approximate solutions to obtain an exact solution.

Step 2: "Almost solution" family for the special (degenerate) case, $\varepsilon = 0$. We will to construct a particular family of functions $[-1, 1] \ni \beta \to w_{0,\beta} \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{0\})$, which are odd in x and such that $\beta \to w_{0,\beta}$ is C^1 in β. Further, the $w_{0,\beta}$ asymptotically, as $x \to \pm\infty$, converge to a heteroclinic travelling wave solution. We do not require them to be close to a solution near the dislocation, $x = 0$.

For $\varepsilon = 0$, such a family $w_{0,\beta}$ is obtained by choosing

$$w_{0,\beta} = u_p + B\beta u_o \tag{15}$$

for some small constant $B > 0$, where $u_p \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{0\})$ is the particular odd solution to (13) of [13] discussed in Step 1; the odd function $u_o \in C^4(\mathbb{R})$ vanishes in a neighbourhood of 0 and satisfies for some $\nu < 0$

$$u_o - u^\pm_{o,\infty} \in E^\nu_4(\mathbb{R} \setminus [-1, 1], \mathbb{R}) \text{ with } u^\pm_{o,\infty}(x) := \mathrm{sgn}(x) \cos(k_0 x). \tag{16}$$

As in Step 1, there is no work to be done; indeed, the existence of such a function u_o is, as in [3], obvious: choose any odd smooth u_0 that vanishes in a neighbourhood of 0 and that is equal to $\operatorname{sgn}(x)\cos(k_0 x)$ outside another, larger, neighbourhood.

Step 3: "Almost solution" family for $\varepsilon > 0$. As indicated before, we shall build from $w_{0,\beta}$ a particular family of functions $[-1,1] \ni \beta \to w_\beta \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{0\})$ which are odd in x and such that $[-1,1] \ni \beta \to w_\beta \in W^{2,\infty}(\mathbb{R})$ is C^1 in β, for $\varepsilon > 0$ small enough. Additionally, w_β will satisfy $\operatorname{sgn}(w_\beta(x)) = \operatorname{sgn}(x)$ on $\mathbb{R}$, $w'_\beta(0) > 0$, $w_\beta(x)$ will tend to a positive periodic solution $w^+_{\beta,\infty}$ to the equation $Lv - \alpha\psi'(v) = 0$ as $x \to \infty$ and $w_\beta(x)$ will tend to a negative periodic solution $w^-_{\beta,\infty}$ as $x \to -\infty$. For simplicity, we do not note explicitly the dependence of w_β on ε.

For small $\varepsilon > 0$, to obtain w_β from $w_{0,\beta}$, we use, with some modifications, the centre manifold theory developed by Iooss and Kirchgässner [11], in our case applied near the constant solutions ± 1. Moreover $w_\beta = u_p$ in a neighbourhood of 0 independent of $\beta \in [-1,1]$ and small $\varepsilon > 0$. This centre manifold argument is presented in Section 3. We do not perform a normal form reduction as Iooss and Kirchgässner, but instead parametrise the centre manifold and obtain a "homotopy" that allows us to construct approximate solutions w_β.

Step 4. Existence proof. We shall then study the existence of $\beta \in [-1,1]$ and a "corrector function" r in an appropriate space of bounded functions such that $w_\beta - r$ is a solution to the equation

$$c^2\Big(w_\beta - r\Big)'' - \Delta_D\Big(w_\beta - r\Big) + \alpha\Big(w_\beta - r\Big) - \alpha\psi'\Big(w_\beta - r\Big) = 0. \tag{17}$$

The outline of the remaining arguments is as follows. In Section 3, we will prove the existence of the family w_β of "approximate" solutions used in Step 3; the argument relies on centre manifold theory. Properties of this family of functions are established in Section 4. Section 5 contains the fixed point argument used in Step 4 and thus finishes the proof.

3 Construction of asymptotic wave trains

Since the first two steps of the proof strategy outlined in the previous section entirely rely on existing results, we now focus on Step 3. Specifically, we construct a family w_β of wave trains which have asymptotically the correct behaviour, in the sense that they solve (4) as $x \to \pm\infty$. This is the key step in the argument, as the anharmonicity of the wells of the on-site potential is now crucial. We use centre manifold theory. Note that w_β are only approximate solutions to (4); for $\beta \in \{-1,1\}$ they will typically differ significantly from solutions near the dislocation site $x = 0$, and be only asymptotically correct for large values of $|x|$ (see the third part of Proposition 4.1).

In Section 5, we prove the existence of a corrector r such that $w_\beta - r$ solves (4) or, equivalently, (17). We remark that the symmetry of the problem is important here. In essence, w_β glues together two wave trains, one as $x \to -\infty$ oscillating in the well of ψ centred at -1, and one as $x \to \infty$ oscillating in the well centred at 1. In principle, there could be a phase shift within the oscillations of the solution as $x \to \infty$ and the point-symmetric continuation of the solution as $x \to -\infty$ from the origin; then the corrector could not be in L^2, as it would have to shift the oscillations on a half line. Here the symmetry of the problem means that such a phase shift does not occur.

Let us now state the main result of this section, the proof of which is postponed to the end (Subsection 3.2). Throughout this section, the standing assumptions are those made in Theorem 2.1.

The main aim is to prove the existence of the "approximate" solutions w_β proposed in Step 3 in the previous section. This is a nontrivial problem, as we require these functions to be asymptotic to periodic solutions (Item 4) in the following theorem). The results establishes the existence of a function H_1 which maps $w_{0,\beta}$ and its derivative to w_β, in a pointwise manner. We recall the definition (15) of $w_{0,\beta}$, and that u_p is the solution to (13).

Theorem 3.1. *For all $\varepsilon > 0$ small enough, there exists $H_1 \in C^4(\mathbb{R}^2)$ and a period map $\widetilde{\mathcal{P}} \in C^4([-1,1],(0,\infty))$, both depending on ε, such that*

1) $H_1(u,v) - u \to 0$ in $W^{4,\infty}(\mathbb{R}^2)$ as $\varepsilon \to 0$.

2) $H_1(u,v)$ is odd in u.

3) $H_1(u,v) = u$ on $(-\varepsilon_0/2, \varepsilon_0/2) \times \mathbb{R}$ for some $\varepsilon_0 > 0$ independent of ε.

4) With $\widetilde{x} := \dfrac{2\pi x}{\widetilde{\mathcal{P}}(\beta) k_0}$, the function $w_\beta(x) := H_1\Big(w_{0,\beta}(\widetilde{x}), w_{0,\beta}'(\widetilde{x})\Big)$ is asymptotic to a positive periodic solution to (4) of period $\widetilde{\mathcal{P}}(\beta)$ as $x \to +\infty$.

5) Furthermore, $sgn(w_\beta(x)) = sgn(x)$ for all $x \in \mathbb{R}$, $w_\beta(x) = u_p(\widetilde{x})$ in a neighbourhood of 0 independent of $\beta \in [-1,1]$ and small $\varepsilon > 0$, and $w_\beta'(0) = \dfrac{2\pi}{\widetilde{\mathcal{P}}(\beta) k_0} u_p'(0) > 0$.

6) $\widetilde{\mathcal{P}}(\beta) \to 2\pi/k_0$ and $\frac{d}{d\beta}\widetilde{\mathcal{P}}(\beta) \to 0$ uniformly in $\beta \in [-1,1]$ as $\varepsilon \to 0$.

Remark. Additional assumptions on higher order derivatives of the map $u \to \psi_\varepsilon(u)$ for $|u| \geq \varepsilon$ would allow higher-order convergence in claim 1). Note, however, that the third part implies that H_1 is not only C^4 but even smooth on $(-\varepsilon_0/2, \varepsilon_0/2) \times \mathbb{R}$.

The proof of Theorem 3.1 is given in Subsection 3.2.

3.1 Centre manifold analysis

As preparation, we perform a centre manifold analysis, following closely [11] (as excellent other source on the centre manifold approach for lattice systems, we refer the reader to [12]). Small modifications are required, since the analysis in [11] is local, while we need a global result, as dislocation waves have large oscillations. A minor change is that in [11], it is assumed that the equilibrium is in 0. Obviously, by adding, a constant the equilibrium can be shifted to 1 (or -1), the wells of the on-site potential ψ_ε. Indeed, this is possible since ψ_ε'' is under control around the equilibria 1 and -1 when $\varepsilon \to 0$, by (8).

In [11], the governing equation is written as

$$\ddot{u}(t) + \tau^2 V'(u(t)) = \gamma\tau^2[u(t-1) - 2u(t) + u(t+1)], \tag{18}$$

where $u(t)$ stands for $u(x)$, and $\tau, \gamma > 0$ are given by

$$\tau^2 := \alpha/c^2, \qquad \gamma := 1/\alpha. \tag{19}$$

We follow the notation of [11] for a while, partially since it is convenient to have a potential V with a single minimum at the origin. Later we will translate the results to our setting and notation, and thus transplant the results to the two different equilibria ± 1. The potential V is assumed to be of class C^{m+1} for some $m \geq 1$ (later, we will only require $m = 4$), with $V''(0) = 1$. Note that $(u - \operatorname{sgn}(u))' = 1$ at $u = 1$. Equation (18) is then rewritten as

$$\partial_t U = L_{\gamma,\tau} U + M_\tau(U) \tag{20}$$

with

$$U(t)(s) = (u(t), v(t), W(t,s))^T, \quad W(t,0) = u(t), \quad s \in [-1,1],$$
$$L_{\gamma,\tau} = \begin{pmatrix} 0 & 1 & 0 \\ -\tau^2(1+2\gamma) & 0 & \gamma\tau^2(\delta^1 + \delta^{-1}) \\ 0 & 0 & \partial_s \end{pmatrix},$$

where $\delta^{\pm 1}$ stands for the evaluation at ± 1, and

$$M_\tau(U) = \tau^2(0, u - V'(u), 0)^T \quad \text{with} \quad U = (u, v, W(\cdot))^T.$$

As in [11], let $\mathbb{H}$ and $\mathbb{D}$ be Banach spaces for $U = (u, v, W(\cdot))^T$,

$$\mathbb{H} := \mathbb{R}^2 \times C[-1,1]$$

and

$$\mathbb{D} := \{U = (u, v, W(\cdot))^T \in \mathbb{R}^2 \times C^1[-1,1] : W(0) = u\},$$

both equipped with the maximum norm. The map M_τ is $C^m(\mathbb{D}, \mathbb{D})$.

Let the reflection S in $\mathbb{H}$ be defined by

$$S(u, v, W)^T := (u, -v, W \circ \rho)^T, \text{ with } \rho(s) := -s,$$

and note that $L_{\gamma,\tau}$ and M_τ anticommute with S ("reversibility").

We denote by Δ_0 the set of pairs (γ, τ) such that the part of the spectrum of $L_{\gamma,\tau}$ that lies in $i\mathbb{R}$ contains only one pair of simple eigenvalues (they have to sum up to 0, thanks to reversibility).

In our setting, $(\gamma,\tau)\in\Delta_0$, since for $c\le 1$ and close to 1 the dispersion function D has exactly two roots, as shown in Section 2, and the roots are not degenerate. We denote by P_1 the projection onto the two-dimensional eigenspace related to the two eigenvalues in $i\mathbb{R}$ and set $Q_h := I - P_1$.

Iooss and Kirchgässner refer to Theorem 3 in [28] to prove their theorem about the existence of a *local* centre manifold (that is, under less restrictive conditions on $\mathrm{id} - V'$ than in the Theorem 3.2 below, there is a neighbourhood Ω of 0 in $\mathbb{D}$ such that the result holds for $\tilde U_c\colon \mathbb{R}\to\Omega_c$ rather than $\mathbb{D}_c$ in claim 1) and $\tilde U\colon\mathbb{R}\to\Omega$ in claim 2). Instead, by referring to Theorem 2 in [28], one gets in the same way the following theorem (*global* centre manifold)[1].

Theorem 3.2. *Given $(\gamma,\tau)\in\Delta_0$, assume that $\mathrm{id}-V'\in C^m(\mathbb{R})$, $\mathrm{id}-V'$ is Lipschitz continuous and that the Lipschitz constant is small enough (in a way that depends in particular on m).*

Then there exists a mapping $h\in C_b^m(\mathbb{D}_c,\mathbb{D}_h)$ (all derivatives up to order m are bounded on $\mathbb{D}_c$), where $\mathbb{D}_c := P_1\mathbb{D}$ and $\mathbb{D}_h := Q_h\mathbb{D}$, and a constant $p_0>0$ such that the following is true.

1) *If $\widetilde U_c\colon\mathbb{R}\to\mathbb{D}_c$ is a solution of (21),*

$$\partial_t U_c = L_{\gamma,\tau}U_c + P_1 M_\tau[U_c + h(U_c)], \tag{21}$$

then $\widetilde U = \widetilde U_c + h(\widetilde U_c)$ solves (20).

2) *If $\widetilde U$ solves (20) for all $t\in\mathbb{R}$ and $\|e^{-\eta|\cdot|}\widetilde U\|_{L^\infty(\mathbb{R})}<\infty$ for some η in $(0,p_0)$, then*

$$\widetilde U_h(t) = h(\widetilde U_c(t)),\quad t\in\mathbb{R},$$

holds with $\widetilde U_c = P_1\widetilde U$ and $\widetilde U_h = Q_h\widetilde U$, and $\widetilde U_c(t)$ solves (21).

This global aspect is relevant to our setting, since the centre manifold theory will be applied in large neighbourhoods of the equilibria 1 and -1 (but small enough to exclude a small neighbourhood of the origin, where the convexity of the on-site potential fails). However, in the present abstract setting, the equilibrium is near the origin, in the sense that if $M_\tau=0$, then $h(0)=0$ and $U=0$ is the unique equilibrium.

Inspecting the proof of Theorem 2 in [28], one sees that the norm of $h\in C_b^m(\mathbb{D}_c,\mathbb{D}_h)$ tends to 0 when the norm of $M_\tau\in C_b^m(\mathbb{D},\mathbb{D})$ tends to 0[2]. Moreover, h commutes with the reversibility operator S, so that the reduced equation (21) is reversible (as shown at the end of Section 2.2 in [28]).

[1] We apply Theorem 2 in [28] when $g\in C_b^m(X;Y)$ (with the notations g,X,Y as in [28]), which makes the proof in [28] shorter. Also, still with the notations of [28], $Y=X$ in our setting. The assumptions of Theorem 2 in [28] are checked for completeness in Appendix B, following the ideas in [11].

[2] As explained on bottom of page 131 in [11], the derivatives of Ψ (in the notations of [11]) can be calculated by formal differentiations of the identity (11) in [11]. This gives estimates of the norms of the derivatives of Ψ in terms of the norms of the derivatives of g (still in the notations of [11]).

When $M_\tau \equiv 0$ and $h \equiv 0$, then

$$\mathbb{D}_c = \{(\delta_1, \delta_2 k_0, \delta_1 \cos(k_0 \cdot) + \delta_2 \sin(k_0 \cdot)) : \delta_1, \delta_2 \in \mathbb{R}\},$$

where $k_0 > 0$ is such that $\pm i k_0$ is in the spectrum of $L_{\gamma,\tau}$ (and there are no other purely imaginary values in the spectrum). A simple computation shows that ik is an eigenvalue with $k \in \mathbb{R}\backslash\{0\}$ if and only if $-\tau^2(1+2\gamma) + \gamma\tau^2 2\cos(k) = -k^2$. When $h = 0$, the two-dimensional linear space $\mathbb{D}_c$ is filled by 0 and the orbits of a smooth one-parameter family of reversible periodic solution

$$t \to U_a(t) := (a\cos(k_0 t), -a k_0 \sin(k_0 t), a\cos(k_0(t + \cdot))\,), \tag{22}$$

with $a > 0$ being the amplitude. So, in essence the centre space is parametrised by the amplitude a. Each of these periodic solutions meets the *reversibility line*

$$\{u_c \in \mathbb{D}_c : S u_c = u_c\} = \{(\delta_1, 0, \delta_1 \cos(k_0 \cdot)) : \delta_1 \in \mathbb{R}\}$$

twice in one of its periods (at $t = 0$ and $t = \pi/k_0$ in the period $[0, 2\pi/k_0)$). The intersection with the reversibility line is transverse: at $t = 0$ (say)

$$\begin{aligned} L_{\gamma,\tau}(a, 0, a\cos(k_0 \cdot)) &= (0, -a k_0^2, -a k_0 \sin(k_0 \cdot)) \\ &\neq (0, a k_0^2, a k_0 \sin(k_0 \cdot)) = S L_{\gamma,\tau}(a, 0, a\cos(k_0 \cdot)) \end{aligned}$$

for all $a > 0$. Let us restrict the amplitude parameter a to any fixed compact interval $[a_1, a_2] \subset (0, \infty)$ with $a_1 < a_2$.

Iooss and Kirchgässner proceed by carrying out a normal form analysis. We proceed differently and give a parametrisation of the centre manifold, with the amplitude a and the time t being the parametrisation parameters.

Proposition 3.3. *For $(a, t) \in [a_1, a_2] \times \mathbb{R}$, define $G(a, t) \in \mathbb{D}$ as the value at time t of the solution on the centre manifold that starts at time 0 at $U_a(0) + h(U_a(0))$, with h given by Theorem 3.2 and U_a as in (22). Then G is of class C^m and, when h tends to 0 in $C_b^m(\mathbb{D}_c, \mathbb{D}_h)$, the map $(a, t) \to G(a, t) - U_a(t)$ tends to 0 in $C_b^m([a_1, a_2] \times [-t_1, t_1])$ for all $t_1 > 0$.*

Moreover, for all $a \in [a_1, a_2]$, $t \to G(a, t)$ is a reversible periodic solution to (20). The corresponding period $\mathcal{P}_a$ is a C^m-function of $a \in [a_1, a_2]$ and, when h tends to 0 in $C_b^m(\mathbb{D}_c, \mathbb{D}_h)$, the map $a \to \mathcal{P}_a$ tends to the constant map $2\pi/k_0$ in $C_b^m[a_1, a_2] = C^m[a_1, a_2]$.

Proof. The fact that G is C^m with respect to (a, t) relies on standard results on dependence of solutions with respect to parameters in finite dimensional dynamical systems (see, e.g., the remarks at the end of Chapter I, Section 7 in [4]). The dynamics on the centre manifold is indeed finite dimensional, see (21).

To show that $G(a, t) - U_a(t)$ tends to zero as h tends to zero, we argue by contradiction. Suppose that $||h_n||_{C_b^m(\mathbb{D}_c, \mathbb{D}_h)} \leq 2^{-n^2}$ for all $n \geq 0$, while the corresponding $G_n(a, t) - U_a(t)$ does not tend to 0 in the sense above. Introduce a C^m-interpolation $\widetilde{h}(\cdot; \mu)$ of the sequence $\{h_n\}_{n \geq 0}$ such that $0 \leq \mu \leq 1$, $\widetilde{h}(\cdot; 0) = 0$ and $\widetilde{h}(\cdot; 2^{-n}) = h_n$ for all $n \geq 0$.

For $(a,t) \in [a_1,a_2] \times \mathbb{R}$ and $\mu \in [0,1]$, define $\widetilde{G}(a,t;\mu) \in \mathbb{D}$ as the value at time t of the solution on the centre manifold that starts at time 0 at $U_a(0) + \widetilde{h}(U_a(0);\mu)$. Then $\widetilde{G}$ is a C^m-interpolation of the sequence $\{G_n\}_{n\geq 0}$ such that $\widetilde{G}(a,t;0) = U_a(t)$ and $\widetilde{G}(\cdot,\cdot;2^{-n}) = G_n$ for all $n \geq 0$. As $\widetilde{G}$ and all its derivatives up to order m are continuous at $\mu = 0$, $\widetilde{G}(\cdots;2^{-n})$ converges to $\widetilde{G}(\cdots;0)$ in $C_b^m([a_1,a_2]\times[-t_1,t_1])$ for all $t_1 > 0$. This is a contradiction, as we have supposed *ad absurdum* that $G_n(a,t) - U_a(t)$ does not tend to 0.

The period $\mathcal{P}_a$ satisfies the equation $E_2P_1G(a,\mathcal{P}_a) = 0$, where E_2 is the projection on the second real component of a vector in $\mathbb{D}$ (and we recall that P_1 is the projection on $\mathbb{D}_c$). When $h = 0$, $\frac{d}{dt}E_2P_1G(a,t)|_{t=\mathcal{P}_a} = -ak_0^2$. Hence, if $h \in C_b^m(\mathbb{D}_c,\mathbb{D}_h)$ is small enough, $\frac{d}{dt}E_2P_1G(a,t)|_{t=\mathcal{P}_a} \neq 0$ for all $a \in [a_1,a_2]$ and, by the implicit function theorem, $\mathcal{P}_a$ is a C^m-function of a. When $h \equiv 0$, $\mathcal{P}_a$ is equal to $2\pi/k_0$. Hence, still by the implicit function theorem, the map $a \to \mathcal{P}_a$ tends to the constant map $2\pi/k_0$ in $C_b^m[a_1,a_2]$ when h tends to 0 in $C_b^m(\mathbb{D}_c,\mathbb{D}_h)$. □

We now make the main step in establishing the existence of the function H_1 in Theorem 3.1. We remark that first component H_1 of the function H discussed in the following proposition will (with minimal modifications summarised in Proposition 3.5) be a restriction of the function H_1 of Theorem 3.1.

Proposition 3.4. *Let $0 < a_1 < a_2$. There exists a C_b^m map*

$$(u,v) \to H(u,v) = (H_1(u,v), H_2(u,v), H_3(u,v)) \in \mathbb{D}$$

defined for pairs $(u,v) \in \mathbb{R}^2$ which satisfy $a_1 \leq \sqrt{u^2 + k_0^{-2}v^2} \leq a_2$, with the following properties. If $a \in [a_1,a_2]$, the set

$$\Big\{\Big(a\cos(k_0t), -ak_0\sin(k_0t)\Big) : t \in \mathbb{R}\Big\}$$

belongs to the domain of H. The map

$$t \mapsto H\Big(a\cos(2\pi t/\mathcal{P}_a), -ak_0\sin(2\pi t/\mathcal{P}_a)\Big)$$

is a $\mathcal{P}_a$-periodic and reversible solution to (20) *(or, equivalently, its first component solves* (18)*) on the centre manifold. When $||h||_{C_b^m(\mathbb{D}_c,\mathbb{D}_h)}$ tends to 0, the map (H_1,H_2) tends to the identity map in the C^m-norm (on the domain of H), $\mathcal{P}_a \to 2\pi/k_0$ and $\frac{d}{da}\mathcal{P}_a \to 0$ uniformly in $a \in [a_1,a_2]$.*

Proof. Let $0 < a_1 < a_2$. By Proposition 3.3, for $a \in [a_1,a_2]$, the function $t \to G(a,t) \in \mathbb{D}$ is a reversible periodic solution to (20) with period $\mathcal{P}_a > 0$; it can be parametrised by a and

$$\widetilde{t} := 2\pi t/(\mathcal{P}_a k_0) \in \mathbb{R}.$$

In the variable $\widetilde{t}$, the period is independent of a and equal to $2\pi/k_0$. Hence, we obtain a parametrisation of a compact piece of the centre manifold

$$(a,\widetilde{t}) \to \widetilde{H}(a,\widetilde{t}) = (\widetilde{H}_1(a,\widetilde{t}), \widetilde{H}_2(a,\widetilde{t}), \widetilde{H}_3(a,\widetilde{t})) := G(a,t) \in \mathbb{D}$$

for $a_1 \le a \le a_2$ and $\widetilde{t} \in \mathbb{R}$, which is $2\pi/k_0$-periodic and reversible in $\widetilde{t}$, i.e., $\widetilde{H}(a, -\widetilde{t}) = S\widetilde{H}(a, \widetilde{t})$. This piece of centre manifold is invariant and $a \times \mathbb{R}$ is sent to a reversible periodic solution, up to a linear reparametrisation. When $||h||_{C_b^m(\mathbb{D}_c, \mathbb{D}_h)}$ is small, the map is near the map

$$(a, \widetilde{t}) \to U_a(\widetilde{t}) = (a\cos(k_0\widetilde{t}), -ak_0\sin(k_0\widetilde{t}), a\cos(k_0\widetilde{t} + \cdot))$$

and actually equal to this map when $h = 0$.

By Proposition 3.3, the map $a \to \mathcal{P}_a$ tends to the constant map $2\pi/k_0$ in $C_b^m[a_1, a_2]$ when h tends to 0 in $C_b^m(\mathbb{D}_c, \mathbb{D}_h)$. Moreover, $\frac{d}{da}\mathcal{P}_a = 0$ when $h = 0$ (because the period is constant, equal to $2\pi/k_0$) and therefore $\frac{d}{da}\mathcal{P}_a \to 0$ uniformly in $a \in [a_1, a_2]$ as $||h||_{C_b^m(\mathbb{D}_c, \mathbb{D}_h)} \to 0$.

Given $(u, v, W) \in \mathbb{D}_c$ in the range of U_a, a can be recovered by the formula

$$a = a(u, v) = \sqrt{u^2 + k_0^{-2}v^2} \tag{23}$$

and, modulo $2\pi/k_0$, $\widetilde{t} = \widetilde{t}(u, v)$ can be recovered from

$$(\cos(k_0\widetilde{t}), \sin(k_0\widetilde{t})) = (a^{-1}u, -a^{-1}k_0^{-1}v).$$

This gives the desired map $(u, v) \to H(u, v) := \widetilde{H}(a(u, v), \widetilde{t}(u, v))$. $\square$

We now return to our initial notation. Let us focus on the well around 1. To apply the centre manifold theorem with order of differentiability $m = 4$, we redefine $\psi = \psi_\varepsilon$ on $(-\infty, \varepsilon)$ so that $|\psi_\varepsilon'(u) - 1| < C\,\varepsilon$ and (10) holds on $\mathbb{R}$ for all small $\varepsilon > 0$. We then obtain the following proposition as reformulation of Proposition 3.4. Note that $a\cos(k_0 t)$ is replaced by $1 + a\cos(k_0 t)$ to take account of the fact that we are now concerned with the well of ψ centred at 1. Moreover, we revert to writing x instead of t, and write the wave equation (18) again as in (4),

$$c^2u'' - \Delta_D u + \alpha u - \alpha\psi'(u) = 0.$$

Proposition 3.5. *Let $0 < a_1 < a_2 < 1$. For all $\varepsilon > 0$ small enough, there exists a C_b^m map*

$$(u, v) \to H(u, v) = (H_1(u, v), H_2(u, v), H_3(u, v)) \in \mathbb{D}$$

defined for pairs $(u, v) \in \mathbb{R}^2$ satisfying $a_1 \le \sqrt{(u-1)^2 + k_0^{-2}v^2} \le a_2$, with the following properties. If $a \in [a_1, a_2]$, the set

$$\Big\{\Big(1 + a\cos(k_0 x), -ak_0\sin(k_0 x)\Big) : x \in \mathbb{R}\Big\}$$

belongs to the domain of H. The map

$$x \to H\Big(1 + a\cos(2\pi x/\mathcal{P}_a), -ak_0\sin(2\pi x/\mathcal{P}_a)\Big)$$

is a $\mathcal{P}_a$-periodic and reversible solution to (4) on the centre manifold. When $\varepsilon \to 0$, the map (H_1, H_2) tends to the identity map in the C^m-norm (on the domain of H), $\mathcal{P}_a \to 2\pi/k_0$ and $\frac{d}{da}\mathcal{P}_a \to 0$ uniformly in $a \in [a_1, a_2]$.

This proposition in particular establishes the existence of H_1, the first component of H. We will prove in the following subsection that a suitable extension of this function has the properties of the function H_1 claimed in Theorem 3.1.

3.2 Proof of Theorem 3.1

The function H_1 of Proposition 3.5 establishes the existence of wave trains oscillating in the well centred at 1. We now extend this function by symmetry to a smooth function that gives anharmonic wave trains oscillating in the wells at ± 1 as $x \to \pm\infty$.

The map $(u, v) \to H_1(u, v)$ of the last proposition sends the function

$$\mathbb{R} \ni x \to \Big(1 + a\cos(2\pi x/\mathcal{P}_\alpha)\,,\ -ak_0\sin(2\pi x/\mathcal{P}_\alpha)\Big) \tag{24}$$

to a periodic solution to the equation (4) ($a_1 \leq a \leq a_2$). As ε is near 0, $H_1(u, v)$ is near u by Proposition 3.5 and, when $\varepsilon = 0$, $H_1(u, v) = u$. Given $0 < a_1 < a_2 < 1$, let $\varepsilon_0 > 0$ be such that $u > \varepsilon_0$ and $H_1(u, v) > \varepsilon_0$ for all $(u, v) \in \mathbb{R}^2$ with $a_1 \leq \sqrt{(u-1)^2 + k_0^{-2}v^2} \leq a_2$. Since $\psi \in C^3$, we can assume that H_1 is C^2; moreover H_1 is well-defined on a compact convex subset of $(\varepsilon_0, \infty) \times \mathbb{R}$ with non-empty interior. We can then extend $H_1 - u$ in a C^4 way on $\mathbb{R}^2$, such that $H_1 - u$ is small in $W^{4,\infty}(\mathbb{R}^2)$; see [25, Paragraph VI 2.3]. The extension can be chosen such that $H_1(u, v)$ is odd in u and that $H_1(u, v) = u$ on $(-\varepsilon_0/2, \varepsilon_0/2) \times \mathbb{R}$. Remembering that ψ' is odd, the analysis around the well 1 as $x \to \infty$ can therefore be transferred to the well -1 as $x \to -\infty$. This establishes claims 1)–3) of Theorem 3.1.

We now turn to the proof of claims 4) and 6) of this theorem. For $x \in \mathbb{R}$, $w_\beta(x)$ has been defined there as

$$w_\beta(x) = H_1\Big(w_{0,\beta}\Big(2\pi x/(\mathcal{P}_a k_0)\Big),\, w'_{0,\beta}\Big(2\pi x/(\mathcal{P}_a k_0)\Big)\Big),$$

where $w_{0,\beta}$ is given by (15) and $\mathcal{P}_a > 0$ is the period corresponding to

$$a = \lim_{x \to +\infty} \sqrt{(w_{0,\beta}(x) - 1)^2 + k_0^{-2} w'_{0,\beta}(x)^2}\,, \tag{25}$$

analogously to (23). The constant 1 has been subtracted from $w_{0,\beta}(x)$ since the analysis is carried out around the constant solution 1 when $x \to +\infty$.

As a is a function of β, so is $\mathcal{P}_a$, and we set $\widetilde{\mathcal{P}}(\beta) = \mathcal{P}_a$. Let us go back to the definition of $w_{0,\beta}$ in (15),

$$w_{0,\beta} = u_p + B\beta u_o,$$

where $B > 0$, u_p is the particular odd solution of (13) found in [13] for $\varepsilon = 0$, and the odd function u_o satisfies (16) and vanishes in a neighbourhood of 0. In [13], it is shown that $u_p'(0) > 0$, $\mathrm{sgn}(u_p(x)) = \mathrm{sgn}(x)$ on $\mathbb{R}$ and u_p converges exponentially to

$$u^\pm_{p,\infty}(x) = \pm\left\{1 - \frac{c^2k_0^2 - 2}{c^2k_0^2 - k_0}\cos(k_0 x)\right\} \tag{26}$$

as $x \to \pm\infty$. In fact, as $x \to +\infty$ (resp. $x \to -\infty$), the rate of convergence is exponential of the type $e^{-|\nu x|}$ in the sense that the difference and its three first derivatives are bounded on $\mathbb{R}$ when multiplied by $e^{|\nu x|}$. Moreover the parameter $\nu < 0$ can be assumed to satisfy $|\nu| < p_0$, with p_0 as in Theorem 3.2. Furthermore, it holds that $0 < \inf u^+_{p,\infty} < \sup u_{p,\infty} < 1$. We choose $0 < a_1 < a_2 < 1$ such that $a_1 < \inf u^+_{p,\infty} < \sup u_{p,\infty} < a_2$. Then $B > 0$ in the definition of $w_{0,\beta}$ is chosen small enough so that $a_1 < \inf u^+_{p,\infty} - B < \sup u_{p,\infty} + B < a_2$. Let us ignore for the moment issues of convergence and study the image of $u^\pm_{p,\infty} + B\beta u_o$ and its derivative under H_1, rather than the image of $w_{0,\beta} = u_p + B\beta u_o$ and its derivative. Note that

$$\left(u^\pm_{p,\infty}(\widetilde{x}) + B\beta u_o(\widetilde{x}), {u^\pm_{p,\infty}}'(\widetilde{x}) + B\beta u_o'(\widetilde{x})\right)$$

is of the form (24) for small enough constant B; thus by Proposition 3.5 for all $|\beta| \le 1$

$$x \to H_1\left(u^\pm_{p,\infty}(\widetilde{x}) \pm B\beta\cos(k_0\widetilde{x}), {u^\pm_{p,\infty}}'(\widetilde{x}) \mp k_0 B\beta \sin(k_0\widetilde{x})\right), \qquad (27)$$

with $\widetilde{x} = 2\pi x/(\mathcal{P}_a k_0)$ is a periodic solution to (4), where the period is $\mathcal{P}_a > 0$ and (see (25))

$$a = \left| B\beta - \frac{c^2k_0^2 - 2}{c^2k_0^2 - k_0} \right|.$$

We are thus left with studying the convergence of $w_{0,\beta}$, that is, the convergence of u_p to $u_{p,\infty}$. In [13], u_p is shown to be of the form $u_p = \widetilde{u}_p - r$, with the following properties.

1) The odd function $\widetilde{u}_p$ and the Fourier transform $\widehat{r}$ of r are explicitly given.

2) The function $\widetilde{u}_p \in W^{2,\infty}(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{0\})$ converges exponentially to $u^\pm_{p,\infty}(x)$ as $x \to \pm\infty$, with corresponding exponential convergence of its four first derivatives.

3) $L\widetilde{u}_p - \alpha\operatorname{sgn}(x)$ is continuous at $x = 0$.

4) The Fourier transform $k \to \widehat{r}(k)$ is smooth and decays with all its derivatives to 0 at $\pm\infty$ at least as $|k|^{-5}$.

5) The identity $\operatorname{sgn}(\widetilde{u}_p(x)) = \operatorname{sgn}(u_p(x)) = \operatorname{sgn}(x)$ holds on $\mathbb{R}$.

As $Lr = L\widetilde{u}_p - \alpha\operatorname{sgn}(x)$ decays exponentially to 0, so do r and r' by Proposition A.2. As $c^2r'' = \Delta_D r - \alpha r + L\widetilde{u}_p - \alpha\operatorname{sgn}(x)$ and the two first derivatives of $L\widetilde{u}_p - \alpha\operatorname{sgn}(x)$ decays exponentially, so do r'', r''' and $r^{(4)}$. As $u_p'' = \Delta_D u_p - \alpha u_p + \alpha\operatorname{sgn}(x)$, we get that $u_p \in C^3(\mathbb{R}\backslash\{0\}) \cap W^{3,\infty}(0,\infty)$ and that u_p is piecewise C^3 on $\mathbb{R}$.

This decay, in combination with property 2) and the fact that (27) defines, as just shown, a periodic solution to (4), establishes claims 4) and 6) of Theorem 3.1. Finally, claim 5) of the theorem follows immediately from the fact that $\operatorname{sgn}(u_p(x)) = \operatorname{sgn}(x)$ on $\mathbb{R}$ with $u_p'(0) > 0$, as shown in [13]; then $\operatorname{sgn}(w_{0,\beta}(x)) = \operatorname{sgn}(x)$ on $\mathbb{R}$ with $w_{0,\beta}'(0) > 0$ by (15); the result follows since by claim 2) $H(u,v)$ is odd in u and by claim 3) $H_1(u,v) = u$. $\square$

4 Properties of the family w_β

In this section, we establish various properties of the family w_β which will be used in the fixed point argument in Section 5 to complete the proof of Theorem 2.1. Throughout this section, w_β will be as defined in Theorem 3.1. Let $F_0^\nu(\mathbb{R}) := \{f \in L^\infty(\mathbb{R}) : e^{-\nu|x|} f \in L^\infty(\mathbb{R})\}$.

Proposition 4.1. *For all $\nu < 0$ close enough to 0, the following holds.*

1) With L defined in (5),

$$Lw_\beta - \alpha\psi'(w_\beta) \in F_0^\nu(\mathbb{R})$$

and is $C^1([-1,1], F_0^\nu(\mathbb{R}))$ as a function of $\beta \in [-1,1]$.

2) The transversality condition

$$\int_{\mathbb{R}} \frac{d}{d\beta}\Big(Lw_\beta - \alpha\psi'(w_\beta)\Big)\sin(k_0 x)dx \neq 0 \tag{28}$$

holds for all $\beta \in [-1,1]$.

3) In addition, we have

$$\lim_{(\beta,\varepsilon)\to 0} \sup\Big\{e^{|\nu x|}\,|(Lw_\beta)(x) - \alpha\psi'(w_\beta(x))| : x \in \mathbb{R}, |w_\beta(x)| \geq \varepsilon\Big\} = 0.$$

In this proposition, the sign convention $\nu < 0$ is chosen to be consistent with the notations in [11, 28]. Moreover, claim 1) ensures that the expression in (28) is well defined. In the proof, we shall see that in fact

$$K_0 := \inf_{\beta\in[-1,1]} \left|\frac{d}{d\beta}\int_{\mathbb{R}}((Lw_\beta - \alpha\psi'(w_\beta))\sin(k_0 x)dx\right| > 0.$$

Proof. In this proof, $B > 0$ is chosen as small and $\nu < 0$ as close to 0 as required (but in a way that is independent of ε small). Let us reconsider the function w_β from Theorem 3.1 and show that we have exponentially attained limits. To this behalf, we write

$$w_\beta(x) = H_1\Big(w_{0,\beta}(\widetilde{x}), w_{0,\beta}'(\widetilde{x})\Big) \quad \text{with} \quad w_{0,\beta} = u_p + B\beta u_o \tag{29}$$

for $x \in \mathbb{R}$ with $\widetilde{x} = 2\pi x/(\widetilde{\mathcal{P}}(\beta)k_0)$ and $\widetilde{\mathcal{P}}(\beta)$ as in Theorem 3.1.

We write $u_{p,\infty}^\pm$, $u_{o,\infty}^\pm$ and $w_{\beta,\infty}^\pm$ for the corresponding asymptotic periodic functions as $x \to \pm\infty$. Let us choose $C_0 > 0$ large enough so that $|w_\beta| \geq \varepsilon$ on $\mathbb{R}\backslash(-C_0\varepsilon, C_0\varepsilon)$ for all $\beta \in [-1,1]$ and all small ε. We recall that $w_{\beta,\infty}^+ \geq \varepsilon$ and $w_{\beta,\infty}^- \leq -\varepsilon$ on $\mathbb{R}$ if $\varepsilon > 0$ is small enough.

We obtain by continuity of H_1

$$w_{\beta,\infty}^\pm(x) = H_1\Big(w_{0,\beta,\infty}^\pm(\widetilde{x}), {w_{0,\beta,\infty}^\pm}'(\widetilde{x})\Big) \quad \text{with} \quad w_{0,\beta,\infty}^\pm = u_{p,\infty}^\pm + B\beta u_{o,\infty}^\pm.$$

To prove claim 1), we first prove an auxiliary statement. Let $d_a^\pm := w_{0,\beta} - w_{0,\beta,\infty}^\pm$, so that by the fundamental theorem of calculus

$$w_\beta(x) - w_{\beta,\infty}^\pm(x) = \int_0^1 d_a^\pm \partial_1 H_1 \left(w_{0,\beta,\infty}^\pm + \sigma d_a^\pm, {w_{0,\beta,\infty}^\pm}' + \sigma {d_a^\pm}'\right) + {d_a^\pm}' \partial_2 H_1 \left(w_{0,\beta,\infty}^\pm + \sigma d_a^\pm, {w_{0,\beta,\infty}^\pm}' + \sigma {d_a^\pm}'\right) d\sigma, \quad (30)$$

where the functions in the arguments of H_1 are evaluated at $\widetilde{x}$. This expression is exponentially decaying as $x \to \pm\infty$ since $d_a^\pm$ and ${d_a^\pm}'$ decay exponentially, by (26) and (16). The fact that left-hand side is evaluated at x and the right-hand side at $\widetilde{x}$ is not a problem since we can decrease $|\nu|$. We also get that $\left(w_\beta(x) - w_{\beta,\infty}^\pm(x)\right)'$ and $\left(w_\beta(x) - w_{\beta,\infty}^\pm(x)\right)''$ are exponentially decaying. Here we use that H_1 is of class C^3. These estimates are used to estimate $L(w_\beta - w_{\beta,\infty}^\pm)$ below.

As $Lw_{\beta,\infty}^\pm - \alpha\psi'(w_{\beta,\infty}^\pm) = 0$ by Theorem 3.1, we find that

$$\begin{aligned} Lw_\beta - \alpha\psi'(w_\beta) &= L(w_\beta - w_{\beta,\infty}^\pm) - \alpha\left(\psi'(w_\beta) - \psi'(w_{\beta,\infty}^\pm)\right) \\ &= L(w_\beta - w_{\beta,\infty}^\pm) - \alpha\int_0^1 \psi''\left(w_{\beta,\infty}^\pm + \sigma(w_\beta - w_{\beta,\infty}^\pm)\right) d\sigma \cdot (w_\beta - w_{\beta,\infty}^\pm), \quad (31)\end{aligned}$$

and both terms on the right are also exponentially decaying as $x \to \pm\infty$ by the exponential bound on (30) just established (note that, if $\varepsilon > 0$ is small enough, $|w_\beta|, |w_{\beta,\infty}^\pm| \in [\varepsilon, \infty)$ for all $|x| \geq C_0\varepsilon$ and thus $\left|\psi''\left(w_{\beta,\infty}^\pm + \sigma(w_\beta - w_{\beta,\infty}^\pm)\right)\right| \leq C\ \varepsilon$ (uniformly in β and $\sigma \in [0,1]$). By continuity of ψ' and the other expressions involved, $Lw_\beta - \alpha\psi'(w_\beta) \in L^\infty[-C_0\varepsilon, C_0\varepsilon]$ (remember that $w_\beta \in C^2(\mathbb{R}\backslash\{0\}) \cap W^{2,\infty}(\mathbb{R})$). These arguments prove the first part of claim 1), $Lw_\beta - \alpha\psi'(w_\beta) \in F_0^\nu(\mathbb{R})$.

To establish that this expression is $C^1([-1,1], F_0^\nu(\mathbb{R}))$ as a function of β, we give an argument in three steps.

Step 1. First note that for fixed small $\varepsilon > 0$, the map

$$\beta \to Y_\beta := Lw_\beta - \alpha\psi'(w_\beta)$$

is C^1 in β if Y_β is restricted to any bounded interval (x_0, x_1) and the target space is endowed with the norm of $L^\infty(x_0, x_1)$. Indeed, w_β is obtained from β by composition of C^2 maps in β, u_p, u_o, u_p', u_o' and the change of variables $x \to \widetilde{x}$. Because of property 3) of Theorem 3.1, $u_p'(x)$ is not involved in the definition of w_β for x near 0. However, $u_p(x)$ is involved for all $x \in \mathbb{R}$ (and u_p is C^1). Again because of property 3) of Theorem 3.1, the less regular term Lw_β is related to $L\widetilde{v}_p$, where $\widetilde{v}_p(x) := u_p(\widetilde{x})$ and $\widetilde{x} = 2\pi x/(\widetilde{P}(\beta)k_0)$. As u_p is $C^3(\mathbb{R}\backslash\{0\})$ and piecewise C^3, the C^1 regularity in β of $L\widetilde{v}_p$ follows in $L^\infty(x_0, x_1)$.

Step 2. Hence it remains to check that $\beta \to Y_\beta$ is C^1 if the target space is endowed with the norm of $E_0^\nu(\mathbb{R}\backslash[-C_0\varepsilon, C_0\varepsilon], \mathbb{R})$. Due to the previous argument, with x_0 negative such that $-x_0$ and x_1 are arbitrarily large, the claim is proved if Y_β and $(d/d\beta)Y_\beta$ are bounded in $E_0^{\widetilde{\nu}}(\mathbb{R}\backslash[-C_0\varepsilon, C_0\varepsilon], \mathbb{R})$, uniformly in β, where $\widetilde{\nu} < \nu < 0$. We observe that the estimate on Y_β in (31) is uniform in β, so it remains to analyse the derivative with respect to β in the final step; note that (31) and Step 3. establish these two properties for ν, so the claim follows for slightly smaller $|\nu|$.

Step 3. We recall that $\widetilde{\mathcal{P}}'(\beta) \to 0$ uniformly in β as $\varepsilon \to 0$ (see Theorem 3.1). We thus obtain in analogy to (30) and (31) that $\frac{d}{d\beta}\left(w_\beta - w^{\pm}_{\beta,\infty}\right)$ and $\frac{d}{d\beta}\left(Lw_\beta - \alpha\psi'(w_\beta)\right)$ are exponentially decaying as $x \to \pm\infty$. Here we use that H_1 is of class C^4.

This shows that, after decreasing $|\nu|$ if necessary,

$$\frac{d}{d\beta}\left(Lw_\beta - \alpha\psi'(w_\beta)\right) \in E_0^\nu(\mathbb{R} \setminus [-C_0\varepsilon, C_0\varepsilon], \mathbb{R})$$

with uniform bounds in $\beta \in [-1, 1]$ and small $\varepsilon > 0$.

We move on to claim 2), the transversality relation. Remember that $H_1(u, v)$ tends to u, $\partial_1 H_1(u, v)$ tends to 1 and $\partial_2 H_1(u, v)$ tends to 0 as $\varepsilon \to 0$ by Theorem 3.1. Using these properties and again the fact that $\widetilde{\mathcal{P}}'(\beta) \to 0$ uniformly in β as $\varepsilon \to 0$, we obtain

$$\begin{aligned}\frac{d}{d\beta}(Lw_\beta - \alpha\psi'(w_\beta)) = (L - \alpha\psi''(w_\beta)I)\circ \\ \Big[\partial_1 H_1(\cdot,\cdot)\left(Bu_o(\widetilde{x}) + \left\{u_p'(\widetilde{x}) + B\beta u_o'(\widetilde{x})\right\}\widetilde{x}(-\widetilde{\mathcal{P}}'(\beta)/\widetilde{\mathcal{P}}(\beta))\right) \\ + \partial_2 H_1(\cdot,\cdot)\Big(Bu_o'(\widetilde{x}) + \{u_p''(\widetilde{x}) + B\beta u_o''(\widetilde{x})\}\widetilde{x}(-\widetilde{\mathcal{P}}'(\beta)/\widetilde{\mathcal{P}}(\beta))\Big)\Big]\end{aligned}$$

converges to BLu_o in $L^\infty_{\text{loc}}(\mathbb{R}\backslash\{0\})$ uniformly in $\beta \in [-1, 1]$ as $\varepsilon \to 0$.

Remember also the hypothesis $|\psi_\varepsilon''(u)| \le 2\varepsilon^{-1}$ for all $|u| < \varepsilon$, which leads to $|\psi_\varepsilon''(u)| < C\,\varepsilon^{-1}$ for all $u \in \mathbb{R}$ and for some constant $C > 0$. We thus get

$$\int_{-C_0\varepsilon}^{C_0\varepsilon} |\alpha\psi''(w_\beta(x))\sin(k_0x)|\,dx \le C\,\alpha\varepsilon^{-1}k_0\int_{-C_0\varepsilon}^{C_0\varepsilon}|x|dx \to 0 \text{ as } \varepsilon \to 0,$$

uniformly in $\beta \in [-1, 1]$. Hence the integral

$$\int_{-C_0\varepsilon}^{C_0\varepsilon}\left|\frac{d}{d\beta}\left(Lw_\beta(x) - \alpha\psi'(w_\beta(x))\right)\sin(k_0x)\right|dx$$

converges to 0 uniformly in $\beta \in [-1, 1]$ as $\varepsilon \to 0$, and so does

$$\int_{-C_0\varepsilon}^{C_0\varepsilon}|BLu_o(x)\sin(k_0x)|\,dx \to 0.$$

Therefore

$$\int_{-C_0\varepsilon}^{C_0\varepsilon}\frac{d}{d\beta}\left(Lw_\beta(x) - \alpha\psi'(w_\beta(x))\right)\sin(k_0x) - BLu_o(x)\sin(k_0x)dx \to 0$$

as $\varepsilon \to 0$ and uniformly in $\beta \in [-1,1]$. Note that Proposition A.4 yields

$$\int_{\mathbb{R}} BLu_o(x)\sin(k_0x)dx = B(-2ck_0^2+2) \neq 0.$$

Reducing $B>0$ if needed, we have proved (28) and the remark that follows it.

It remains to show claim claim 3), that is,

$$\lim_{(\beta,\varepsilon)\to 0} \sup\left\{e^{|\nu x|}\,|Lw_\beta(x) - \alpha\psi'(w_\beta(x))| : x\in\mathbb{R}, |w_\beta(x)| \geq \varepsilon\right\} = 0.$$

In (31), we have

$$\left|\chi_{\{|w_\beta|\geq\varepsilon\}}\psi''(w^{\pm}_{\beta,\infty} + \sigma\{w_\beta - w^{\pm}_{\beta,\infty}\})\right| \leq C\,\varepsilon$$

and thus

$$\left\|e^{|\nu x|}\chi_{\{|w_\beta|\geq\varepsilon\}}\cdot(w_\beta - w^{\pm}_{\beta,\infty})\int_0^1 \psi''\left(w^{\pm}_{\beta,\infty} + \sigma\{w_\beta - w^{\pm}_{\beta,\infty}\}\right)d\sigma\right\|_{L^\infty(\mathbb{R})} \leq C\,\varepsilon \to 0$$

as $\varepsilon\to 0$, by taking a smaller $|\nu|$ if needed (see (30)). Moreover, for $\pm$ in (31) replaced by $+$ (respectively $-$),

$$\begin{aligned} e^{|\nu x|}L(w_\beta - w^{\pm}_{\beta,\infty}) = e^{|\nu x|}L\Big(&H_1\Big(u_p(\widetilde{x}) + B\beta u_o(\widetilde{x}), u_p'(\widetilde{x}) + B\beta u_o'(\widetilde{x})\Big) \\ &- H_1\Big(u^{\pm}_{p,\infty}(\widetilde{x}) + B\beta u^{\pm}_{o,\infty}(\widetilde{x}), {u^{\pm}_{p,\infty}}'(\widetilde{x}) + B\beta {u^{\pm}_{o,\infty}}'(\widetilde{x})\Big)\Big) \end{aligned}$$

has its absolute value bounded from above by $C\,e^{|\nu x|}$ on $(0,\infty)$ (respectively $(-\infty,0)$) and converges uniformly on every compact subset of $(0,\infty)$ (respectively $(-\infty,0)$) to

$$e^{|\nu x|}L(u_p - u^{\pm}_{p,\infty}) = e^{|\nu x|}(Lu_p - (\pm\alpha)) = 0$$

as $(\beta,\varepsilon)\to 0$; see (13) and (14). Claim 3) follows by taking a slightly smaller $|\nu|$. □

Motivated by the spaces $E^\nu_m(X)$, we define the solution spaces for the "corrector" r used in Step 4 of the argument, as outlined at the end of Section 2. For $\nu<0$, let

$$E^\nu_{0,odd}(\mathbb{R}) := \{r\in C(\mathbb{R}):\ r \text{ is odd and } e^{|\nu x|}r(x)\in L^\infty(\mathbb{R})\} \tag{32}$$

and

$$E^\nu_{1,odd}(\mathbb{R}) := \{r\in E^\nu_{0,odd}(\mathbb{R})\cap C^1(\mathbb{R}) : e^{|\nu x|}r'(x)\in L^\infty(\mathbb{R})\}. \tag{33}$$

Lemma 4.2. *If $B, \rho, \varepsilon > 0$ are chosen small enough, then for all r in the ball $\overline{B(0,\rho)} \subset E^{\nu}_{1,odd}(\mathbb{R})$*

$$\sup_{\beta\in[-1,1]} \int_{\mathbb{R}} \left|\alpha \left(\psi''(w_\beta) - \psi''(w_\beta - r)\right) \frac{d}{d\beta} w_\beta \sin(k_0 x)\right| dx$$
$$\leq \frac{1}{2} \inf_{\beta\in[-1,1]} \left|\frac{d}{d\beta}\int_{\mathbb{R}} ((Lw_\beta - \alpha\psi'(w_\beta)) \sin(k_0 x) dx\right| := \frac{1}{2} K_0 > 0 \quad (34)$$

and

$$\int_{\mathbb{R}} \left|(c^2 w_0'' - \Delta_D w_0 + \alpha w_0 - \alpha\psi'(w_0 - r)) \sin(k_0 x)\right| dx$$
$$\leq \frac{1}{2} \inf_{\beta\in[-1,1]} \left|\int_{\mathbb{R}} \frac{d}{d\beta} (Lw_\beta - \alpha\psi'(w_\beta)) \sin(k_0 x) dx\right| = \frac{1}{2} K_0. \quad (35)$$

Proof. Choose $B, \rho > 0$ small enough and $C_0 > 0$ large enough (in a way that is independent of small $\varepsilon > 0$) so that $|w_\beta - r| \geq \varepsilon$ on $\mathbb{R}\backslash(-C_0\varepsilon, C_0\varepsilon)$ for all $\beta \in [-1,1]$, $r \in \overline{B(0,\rho)}$ and all small ε. We set $I_1 := \{x : |w_\beta(x)| < \varepsilon\}$, $I_2 := \{x : |w_\beta(x) - r| < \varepsilon\}$ and $I_3 := \mathbb{R} \setminus (I_1 \cup I_2)$. If $B, \rho, \varepsilon > 0$ are chosen small enough, for all $|\beta| \leq 1$ we get from (8) for $j = 1, 2$

$$\int_{I_j} |\psi''(w_\beta) - \psi''(w_\beta - r)| \left|\frac{d}{d\beta} w_\beta\right| |\sin(k_0 x)| dx$$
$$\leq \int_{I_j} C\,\varepsilon^{-1} \left|\frac{d}{d\beta} w_\beta\right| k_0 |x| dx \leq C\,\varepsilon,$$

and for their complement from (10)

$$\begin{aligned}
&\int_{I_3} |\psi''(w_\beta) - \psi''(w_\beta - r)| \left|\frac{d}{d\beta} w_\beta\right| |\sin(k_0 x)| dx \\
&= \int_{I_3} \left|\int_0^1 \psi'''(w_\beta - \sigma r) r d\sigma\right| \left|\frac{d}{d\beta} w_\beta\right| |\sin(k_0 x)| dx \\
&\leq \int_{I_3} C\,\varepsilon |r| \left|\frac{d}{d\beta} w_\beta\right| |\sin(k_0 x)| dx \leq C\,\varepsilon,
\end{aligned}$$

and thus combined

$$\int_{\mathbb{R}} \left|\alpha \left(\psi''(w_\beta) - \psi''(w_\beta - r)\right) \frac{d}{d\beta} w_\beta \sin(k_0 x)\right| dx \leq C\,\varepsilon.$$

As $K_0 > 0$, by (28), this proves (34).

Also, $Lw_0 - \alpha\psi'(w_0) \in E^{\nu}_0(\mathbb{R}\backslash[-1,1])$ with uniform bounds for small enough ε (by taking a smaller $|\nu|$ if necessary, see the comments after (30) and (31)). Further, $Lw_0 - \alpha\psi'(w_0)$ is bounded in $L^\infty(-1,1)$, uniformly for small ε. Moreover, w_0 converges to u_p in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ as ε tends to 0 (see (29)) and $\psi'(w_0(x))$

converges to $\mathrm{sgn}(x)$ for $x \in \mathbb{R} \setminus \{0\}$ as ε tends to 0. Thus

$$\int_{\mathbb{R}} |(Lw_0 - \alpha\psi'(w_0))\sin(k_0 x)|\, dx \to 0 \text{ as } \varepsilon \to 0 \tag{36}$$

since $Lu_p(x) - \alpha\mathrm{sgn}(x) = 0$; we recall that u_p is the solution for the special case of a piecewise quadratic potential [13]. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}} |\psi'(w_0) - \psi'(w_0 - r)|\,|\sin(k_0 x)| dx \\
&= \int_{\mathbb{R}} \left| \int_0^1 \psi''(w_0 - \sigma r) r d\sigma \right| |\sin(k_0 x)| dx \\
&\le \int_{\mathbb{R}} \left(C\,\varepsilon^{-1}\chi_{\{|x|<C_0\varepsilon\}} + C\,\varepsilon\chi_{\{|x|\ge C_0\varepsilon\}} \right) |r|\,|\sin(k_0 x)| dx \\
&\le C \int_{\mathbb{R}} \varepsilon^{-1}\chi_{\{|x|<C_0\varepsilon\}} |r|\, k_0|x| dx + C \int_{\mathbb{R}} \varepsilon\chi_{\{|x|\ge C_0\varepsilon\}} |r|\,|\sin(k_0 x)| dx \le C\,\varepsilon
\end{aligned}$$

and, as a consequence (see (36)),

$$\int_{\mathbb{R}} |(Lw_0 - \alpha\psi'(w_0 - r))\sin(k_0 x)|\, dx \to 0 \tag{37}$$

uniformly in $r \in \overline{B(0,\rho)}$ as $\varepsilon \to 0$. As $K_0 > 0$, this proves (35). $\square$

5 Existence of a heteroclinic connection

In this section, we employ a fixed point argument to prove the existence of a "corrector" r required in Step 4 (introduced at the end of Section 2). We recall the definition of the solution spaces $E^{\nu}_{0,odd}(\mathbb{R})$ and $E^{\nu}_{1,odd}(\mathbb{R})$ in (32) and (33). In addition, we introduce the following Banach space. Let $G^{\nu}_0(\mathbb{R})$ be the Banach space of functions $f \in L^2(\mathbb{R})$ such that

$$||f||_{G^{\nu}_0(\mathbb{R})} := ||e^{-\nu|\cdot|} f||_{L^2(\mathbb{R})} < \infty. \tag{38}$$

Given $\nu < 0$, we would like to find $r \in E^{\nu}_{1,odd}(\mathbb{R}) \cap H^2(\mathbb{R})$ and $\beta \in [-1,1]$ such that $w_\beta - r$ is a solution to equation (17),

$$c^2\Big(w_\beta - r\Big)'' - \Delta_D\Big(w_\beta - r\Big) + \alpha\Big(w_\beta - r\Big) - \alpha\psi'\Big(w_\beta - r\Big) = 0.$$

We shall apply Proposition A.2, the remark following it and Proposition A.3. They address the solution the equation $Lr = Q$ for $r \in E^{\nu}_1(\mathbb{R}) \cap H^2(\mathbb{R})$, where Q is in various spaces of decaying functions and satisfies

$$\int_{\mathbb{R}} Q(x)\sin(k_0 x) dx = \int_{\mathbb{R}} Q(x)\cos(k_0 x) dx = 0.$$

In Proposition A.2, Q belongs to $E^{\nu}_0(\mathbb{R})$ (that is, Q is continuous and the function $e^{|\nu\cdot|}Q$ is bounded); in the remark, Q belongs to $F^{\nu}_0(\mathbb{R})$ (that is, Q and the function $e^{|\nu\cdot|}Q$ are in $L^\infty(\mathbb{R})$) and in Proposition A.3, Q belongs to $G_0^\nu(\mathbb{R})$. Since in the present section Q is odd, only the condition $\int_\mathbb{R} Q(x)\sin(k_0x)dx$ has to be dealt with and $r \in E_{1,odd}^\nu(\mathbb{R}) \cap H^2(\mathbb{R})$.

Lemma 5.1. *The map*

$$(r,\beta) \to \Gamma(r,\beta) := c^2 w_\beta'' - \Delta_D w_\beta + \alpha w_\beta - \alpha\psi'\big(w_\beta - r\big) - \alpha\psi''(w_\beta)r \qquad (39)$$

is well defined as a map $E_{1,odd}^\nu(\mathbb{R}) \times [-1,1] \to F_0^\nu(\mathbb{R})$ *and is of class* C^1*. Moreover,* Γ *is compact.*

Proof. The map $\beta \to \Gamma(0,\beta) = c^2 w_\beta'' - \Delta_D w_\beta + \alpha w_\beta - \alpha\psi'\big(w_\beta\big)$ is well-defined and C^1 by claim 1) of Proposition 4.1. To prove the lemma, we first investigate the difference in the nonlinear terms of the last expression and the one in (39). By the fundamental theorem of calculus and Fubini's theorem, we obtain

$$\begin{aligned}
&\psi'(w_\beta) - \psi'\big(w_\beta - r\big) - \psi''(w_\beta)r = \int_0^1 \left(\psi''(w_\beta - \sigma r) - \psi''(w_\beta)\right) r d\sigma \\
&= -\int_0^1 \left(\int_0^\sigma \psi'''(w_\beta - \tilde\sigma r)r^2 d\tilde\sigma\right) d\sigma = -\int_0^1 \left(\int_{\tilde\sigma}^1 \psi'''(w_\beta - \tilde\sigma r)r^2 d\sigma\right) d\tilde\sigma \\
&= -\int_0^1 (1-\tilde\sigma)\psi'''(w_\beta - \tilde\sigma r)r^2 d\tilde\sigma.
\end{aligned}$$

Setting $e^{|\nu x|}r(x) =: \widetilde{r}(x)$, we thus have to show that the map

$$W^{1,\infty}(\mathbb{R}) \times [-1,1] \ni (\widetilde{r},\beta) \to -e^{-|\nu s|}\int_0^1 (1-\sigma)\psi'''(w_\beta - \sigma e^{-|\nu s|}\widetilde{r})\widetilde{r}^2 d\sigma$$

with range included in $L^\infty(\mathbb{R})$ is C^1 and compact. The first two properties are immediate, and compactness is a consequence of the Arzelà-Ascoli theorem (using the uniform continuity of ψ''' on compact sets and the weight $e^{-|\nu\cdot|}$ in front of the integral). ☐

Proposition 5.2. *Let* $\rho > 0$ *be small enough (see Lemma 4.2). For fixed* r *in* $\overline{B(0,\rho)}$*, the equation*

$$\int_\mathbb{R} \big(c^2 w_\beta'' - \Delta_D w_\beta + \alpha w_\beta - \alpha\psi'\big(w_\beta - r\big)\big)\sin(k_0x)dx = 0$$

can uniquely be solved for β *as a* C^1*-function of* r*,* $\beta = \beta(r)$*. Moreover,* $\beta(r)$ *tends to* 0 *uniformly in* $r \in \overline{B(0,\rho)}$ *as* $\varepsilon \to 0$*.*

Observe that since $\Gamma(r,\beta)$ and $\alpha\psi''(w_\beta)r$ are integrable over $\mathbb{R}$, so is the integrand in the previous equation of Proposition 5.2.

Proof. For fixed r in $\overline{B(0,\rho)}$, let

$$h(\beta) := \int_\mathbb{R} \big(c^2 w_\beta'' - \Delta_D w_\beta + \alpha w_\beta - \alpha\psi'\big(w_\beta - r\big)\big)\sin(k_0x)dx, \quad \beta \in [-1,1].$$

The proposition is a consequence of the fact that, for all $\beta \in [-1,1]$,

$$\begin{aligned}|h'(\beta)| &= \left|\frac{d}{d\beta}\int_{\mathbb{R}} \big(c^2 w_\beta'' - \Delta_D w_\beta + \alpha w_\beta - \alpha\psi'\big(w_\beta - r\big)\big)\sin(k_0 x)dx\right| \\ &\overset{(34)}{\geq} \frac{1}{2}\inf_{\widetilde{\beta}\in[-1,1]}\left|\frac{d}{d\beta}\int_{\mathbb{R}}\Big(c^2 w_{\widetilde{\beta}}'' - \Delta_D w_{\widetilde{\beta}} + \alpha w_{\widetilde{\beta}} - \alpha\psi'(w_{\widetilde{\beta}})\Big)\sin(k_0 x)dx\right| \\ &= \frac{1}{2}K_0,\end{aligned}$$

which implies $\inf_{\beta\in[-1,1]}|h'(\beta)| \overset{(35)}{\geq} |h(0)|$. In turn, this implies $h(\beta) = 0$ for some $\beta \in [-1,1]$, as desired. To this behalf we argue by contradiction and assume for definiteness that $\inf_{\widetilde{\beta}\in[-1,1]} h'(\widetilde{\beta}) \geq h(0) > 0$, so we may set

$$b := -\frac{h(0}{\inf_{\widetilde{\beta}\in[-1,1]} h'(\widetilde{\beta})} \in [-1,0).$$

Then by the intermediate value theorem there exists a $\widetilde{\beta} \in (b,0)$ such that

$$h(b) = h(b) - h(0) + h(0) = h'(\widetilde{\beta})(b-0) + h(0) \leq -h(0) + h(0) = 0.$$

Therefore $h(\beta) = 0$ for some $\beta \in [b,0)$.

More generally, $|\beta| \leq |h(0)|/\inf_{\widetilde{\beta}\in[-1,1]}|h'(\widetilde{\beta})|$ because the infimum is positive (see (34)). Hence $\beta(r)$ tends to 0 uniformly in $r \in \overline{B(0,\rho)}$ as $\varepsilon \to 0$, see (37).

The C^1-dependence of $\beta(r)$ on r is a consequence of the implicit function theorem. $\square$

Proof of Theorem 2.1

The result of this section so far can be formulated as follows. The problem can be written as $c^2 r'' - \Delta_D r + \alpha r = Q$ with $Q := \Gamma(r, \beta(r)) + \alpha\psi''(w_{\beta(r)})r \in F_0^\nu(\mathbb{R})$ odd and $\int_{\mathbb{R}} Q(x)\sin(k_0 x)dx = 0$ by the definition of $\beta(r)$ in Proposition 5.2.

Let $r = L^{-1}Q$ be given by Proposition A.2 (and Remark) in Appendix A applied to Q defined above, so that our problem can be rewritten as

$$r = L^{-1}Q = L^{-1}\Big(\Gamma(r,\beta(r)) + \alpha\psi''(w_{\beta(r)})r\Big). \tag{40}$$

For $\widetilde{\beta} \in [-1,1]$, let $\delta(\widetilde{\beta}, r) \in \mathbb{R}$ be such that

$$\int_{\mathbb{R}}\Big(\alpha\psi''(w_{\widetilde{\beta}})r - \delta(\widetilde{\beta}, r)Lu_o\Big)\sin(k_0 x)dx = 0;$$

Proposition A.4 shows that $\delta(\widetilde{\beta}, r) = \dfrac{\int_{\mathbb{R}}\alpha\psi''(w_{\widetilde{\beta}})r dx}{-2c^2 k_0 + 2}$. Then

$$\int_{\mathbb{R}}\Big(\Gamma(r,\widetilde{\beta}) + \delta(\widetilde{\beta}, r)Lu_o\Big)(x)\sin(k_0 x)dx = 0$$

and

$$r = L^{-1}\Big(\Gamma(r,\widetilde{\beta}) + \delta(\widetilde{\beta},r)Lu_o\Big) + L^{-1}\Big(\alpha\psi''(w_{\widetilde{\beta}})r - \delta(\widetilde{\beta},r)Lu_o\Big).$$

Choose $C_0 > 0$ large enough and $\rho > 0$ small enough so that $|w_\beta - r| \geq \varepsilon$ on $\mathbb{R}\backslash(-C_0\varepsilon, C_0\varepsilon)$ for all $r \in \overline{B(0,\rho)}$, $\beta \in [-1,1]$ and all small ε. Define $\delta_1(\widetilde{\beta}, r)$ and $\delta_2(\widetilde{\beta}, r)$ by

$$\int_{\mathbb{R}} \Big(\alpha\psi''(w_{\widetilde{\beta}})\chi_{\{|x|<C_0\varepsilon\}}r - \delta_1(\widetilde{\beta},r)Lu_o\Big)(x)\sin(k_0x)dx = 0$$

and

$$\int_{\mathbb{R}} \Big(\alpha\psi''(w_{\widetilde{\beta}})\chi_{\{|x|\geq C_0\varepsilon\}}r - \delta_2(\widetilde{\beta},r)Lu_o\Big)(x)\sin(k_0x)dx = 0.$$

Thanks to $|\sin(k_0x)| \leq |k_0x|$,

$$|\psi''(w_{\widetilde{\beta}})| < C\,\varepsilon^{-1} \quad \text{and} \quad |r(x)| \leq C\,|x| \cdot ||r'||_{L^\infty(-C_0\varepsilon,C_0\varepsilon)} \tag{41}$$

on $(-C_0\varepsilon, C_0\varepsilon)$ (because $r(0) = 0$), we get $\delta_1(\widetilde{\beta}, r) = O\Big(\varepsilon^2||r'||_{L^\infty(-C_0\varepsilon,C_0\varepsilon)}\Big)$. Also $|\psi''(w_{\widetilde{\beta}})| < C\ \varepsilon$ on $\mathbb{R}\backslash(-C_0\varepsilon, C_0\varepsilon)$, $\delta_2(\widetilde{\beta}, r) = O\Big(\varepsilon||r||_{L^1(\mathbb{R})}\Big)$ and thus

$$\delta(\widetilde{\beta},r) = \delta_1(\widetilde{\beta},r) + \delta_2(\widetilde{\beta},r) = O\Big(\varepsilon||r||_{E^\nu_{1,odd}(\mathbb{R})}\Big) \tag{42}$$

uniformly in $\widetilde{\beta} \in [-1,1]$. The maps δ_1 and δ_2 are clearly linear in r and, moreover, continuous because of the continuity of the map

$$\widetilde{\beta} \to \psi''(w_{\widetilde{\beta}}) \in L^\infty(\mathbb{R}).$$

Furthermore,

$$\|\alpha\psi''(w_{\widetilde{\beta}} + \sigma r)\chi_{\{|x|\geq C_0\varepsilon\}}r\|_{F^\nu_0(\mathbb{R})} \leq C\,\varepsilon\|r\|_{E^\nu_0(\mathbb{R})} \tag{43}$$

for all $\sigma \in [-1,0]$, and

$$\|\alpha\psi''(w_{\widetilde{\beta}})\chi_{\{|x|<C_0\varepsilon\}}r\|_{G^\nu_0(\mathbb{R})} \leq C\,\varepsilon^{1/2}||r'||_{L^\infty(-C_0\varepsilon,C_0\varepsilon)}\,. \tag{44}$$

See (41). By Proposition A.2,

$$\Big\|L^{-1}\Big(\alpha\psi''(w_{\widetilde{\beta}})\chi_{\{|x|\geq C_0\varepsilon\}}r - \delta_2(\widetilde{\beta},r)Lu_o\Big)\Big\|_{E^\nu_{1,odd}(\mathbb{R})} \leq C\,\varepsilon\|r\|_{E^\nu_0(\mathbb{R})}$$

and, by Proposition A.3,

$$\Big\|L^{-1}\Big(\alpha\psi''(w_{\widetilde{\beta}})\chi_{\{|x|<C_0\varepsilon\}}r - \delta_1(\widetilde{\beta},r)Lu_o\Big)\Big\|_{E^\nu_{1,odd}(\mathbb{R})} \leq \varepsilon^{1/2}O(||r'||_{L^\infty(-C_0\varepsilon,C_0\varepsilon)})$$

uniformly in $\widetilde{\beta} \in [-1,1]$. Hence the linear map

$$r \to r - L^{-1}\Big(\alpha\psi''(w_{\widetilde{\beta}})r - \delta(\widetilde{\beta},r)Lu_o\Big)$$

is invertible on $\overline{B(0,\rho)}$ if ε is small enough. Let us denote the inverse by $\Xi_{\widetilde{\beta}}\colon E^{\nu}_{1,odd}(\mathbb{R}) \to E^{\nu}_{1,odd}(\mathbb{R})$, which is continuous in $\widetilde{\beta}$ when the operator norm is considered. On the other hand, the map

$$r \to L^{-1}\Big(\Gamma(r,\beta(r)) + \delta(\beta(r),r)Lu_o\Big)$$

is completely continuous on $\overline{B(0,\rho)}$ (that is, continuous and compact); see Lemma 5.1. Therefore

$$r \to \Xi_{\beta(r)}\Big(L^{-1}\Big(\Gamma(r,\beta(r)) + \delta(\beta(r),r)Lu_o\Big)\Big)$$

is completely continuous, too. For $\varepsilon > 0$ small enough, it sends $\overline{B(0,\rho)}$ into $B(0,\rho)$. To see this, we refer to Proposition A.3, (42), the inequality

$$\|\Gamma(r,\beta(r))\chi_{\{|x|\geq C_0\varepsilon\}}\|_{G_0^{\widetilde{\nu}}(\mathbb{R})} \leq C\,\|\Gamma(r,\beta(r))\chi_{\{|x|\geq C_0\varepsilon\}}\|_{F_0^{\nu}(\mathbb{R})}$$

for $\nu < \widetilde{\nu} < 0$ (see 38) and use

$$\begin{aligned}
&\|\Gamma(r,\beta(r))\chi_{\{|x|\geq C_0\varepsilon\}}\|_{F_0^{\nu}(\mathbb{R})}\\
&\leq \|\left(Lw_{\beta(r)} - \alpha\psi'\big(w_{\beta(r)} - r\big)\right)\chi_{\{|x|\geq C_0\varepsilon\}}\|_{F_0^{\nu}(\mathbb{R})}\\
&\qquad + \|\alpha\psi''(w_{\beta(r)})r\chi_{\{|x|\geq C_0\varepsilon\}}\|_{F_0^{\nu}(\mathbb{R})}\\
&\overset{(43)}{\leq} \|\left(Lw_{\beta(r)} - \alpha\psi'\big(w_{\beta(r)} - r\big)\right)\chi_{\{|x|\geq C_0\varepsilon\}}\|_{F_0^{\nu}(\mathbb{R})} + C\,\varepsilon||r||_{E^{\nu}_{0,odd}(\mathbb{R})}\\
&\leq C\,\varepsilon||r||_{E^{\nu}_{0,odd}(\mathbb{R})} + \|\left(Lw_{\beta(r)} - \alpha\psi'\big(w_{\beta(r)}\big)\right)\chi_{\{|x|\geq C_0\varepsilon\}}\|_{F_0^{\nu}(\mathbb{R})}\\
&\qquad + \left\|\alpha\int_{-1}^{0}\psi''(w_{\beta(r)} + \sigma r)rd\sigma\chi_{\{|x|\geq C_0\varepsilon\}}\right\|_{F_0^{\nu}(\mathbb{R})} \overset{(43)}{\to} 0
\end{aligned}$$

uniformly in $r \in \overline{B(0,\rho)}$ as ε tends to 0, thanks to the third part of Proposition 4.1 and the fact that $\beta(r)$ tends uniformly to 0 as ε tends to 0 (see Proposition 5.2). We use also that

$$\begin{aligned}
&\left\|\Gamma(r,\beta(r))\chi_{\{|x|<C_0\varepsilon\}}\right\|_{G_0^{\nu}(\mathbb{R})}\\
&\leq \left\|\left(Lw_{\beta(r)} - \alpha\psi'\big(w_{\beta(r)} - r\big)\right)\chi_{\{|x|<C_0\varepsilon\}}\right\|_{G_0^{\nu}(\mathbb{R})}\\
&\qquad + \left\|\alpha\psi''(w_{\beta(r)})r\chi_{\{|x|<C_0\varepsilon\}}\right\|_{G_0^{\nu}(\mathbb{R})}\\
&\overset{(44)}{\leq} \left\|\left(Lw_{\beta(r)} - \alpha\psi'\big(w_{\beta(r)} - r\big)\right)\chi_{\{|x|<C_0\varepsilon\}}\right\|_{G_0^{\nu}(\mathbb{R})} + C\,\varepsilon^{1/2}||r||_{E^{\nu}_{1,odd}(\mathbb{R})}\\
&\leq C\left(\varepsilon^{1/2} + \varepsilon^{1/2}||w'_{\beta(r)} - r'||_{L^{\infty}(\mathbb{R})}\right) + C\,\varepsilon^{1/2}||r||_{E^{\nu}_{1,odd}(\mathbb{R})} \to 0
\end{aligned}$$

uniformly in $r \in \overline{B(0,\rho)}$ as ε tends to 0, since

$$\begin{aligned}
|\psi'(w_{\beta(r)}(x) - r(x))| &\leq ||\psi''||_{L^{\infty}(\mathbb{R})}||w'_{\beta(r)} - r'||_{L^{\infty}(\mathbb{R})}|x|\\
&\leq C\,\varepsilon^{-1}||w'_{\beta(r)} - r'||_{L^{\infty}(\mathbb{R})}|x|.
\end{aligned}$$

Thus the Schauder fixed point theorem gives a solution $r \in \overline{B(0,\rho)}$ to the equation

$$r = \Xi_{\beta(r)}\Big(L^{-1}\Big(\Gamma(r,\beta(r)) + \delta(\beta(r),r)Lu_o\Big)\Big),$$

and $r = L^{-1}Q \in B(0,\rho) \cap H^2_{odd}(\mathbb{R})$ (see Proposition A.3). $\square$

A Tools from Fourier analysis

We begin with a straightforward but useful generalisation of results in [11]. For $\nu \in \mathbb{R}$, $m \in \{0,1,2,\ldots\}$ and a Banach space X, we recall that $E^\nu_m(X)$ is the Banach space of functions $f \in C^m(\mathbb{R},X)$ equipped with the norm (11),

$$||f||_{E^\nu_m(X)} := \max_{0\le j\le m} ||e^{-\nu|\cdot|}f^{(j)}||_{L^\infty(\mathbb{R},X)} < \infty.$$

In the case $X = \mathbb{R}$, this means $f \in E^\nu_m(\mathbb{R})$ if and only if $f \in C^m(\mathbb{R})$ satisfies

$$\max_{0\le j\le m} \sup_{x\in\mathbb{R}} e^{-\nu|x|}|f^{(j)}(x)| < \infty.$$

In the case $X = C[-1,1]$, $f \in E^\nu_m(C[-1,1])$ can be identified with the continuous mapping $(x,s) \mapsto \tilde{f}(x,s) \in \mathbb{R}$, where $\tilde{f}(x,s)$ is the value at $s \in [-1,1]$ of $f(x) \in C[-1,1]$; then $f \in E^\nu_m(C[-1,1])$ if and only if each $\partial_1^j \tilde{f}$ exists and belongs to $C(\mathbb{R}\times[-1,1])$ for $0 \le j \le m$, and

$$\max_{0\le j\le m} \sup_{(x,s)\in\mathbb{R}\times[-1,1]} e^{-\nu|x|}|\partial_1^j \tilde{f}(x,s)| < \infty.$$

In the case $X = C^1[-1,1]$, $f \in E^\nu_0(C^1[-1,1])$ can be identified with $\tilde{f} \in C(\mathbb{R}\times[-1,1])$ such that in addition to the requirements for $X = C[-1,1]$, also $\partial_2 \tilde{f}$ exists and belongs to $C(\mathbb{R}\times[-1,1])$, and

$$\max_{j\in\{0,1\}} \sup_{(x,s)\in\mathbb{R}\times[-1,1]} e^{-\nu|x|}|\partial_2^j \tilde{f}(x,s)| < \infty.$$

Proposition A.1. *Let $p_0 > 0$ and the measurable map $(k,s) \mapsto \widehat{H}(k,s) \in \mathbb{C}$ be defined on its domain*

$$\{(k,s) \in \mathbb{R}\times[-1,1] : Im\, k \in (-p_0,p_0)\}.$$

We assume that, for each $s \in [-1,1]$, the map $k \mapsto \widehat{H}(k,s)$ is analytic in the strip $\{k \in \mathbb{C} : \operatorname{Im} k \in (-p_0,p_0)\}$ and, for all $\delta \in (0,p_0)$, $(1+|k|)|\widehat{H}(k,s)|$ is bounded in $\{(k,s) \in \mathbb{C}\times[-1,1] : \operatorname{Im} k \in [-\delta,\delta]\}$.

Then, for every $s \in [-1,1]$, $\widehat{H}(\cdot,s)\colon \mathbb{R} \to \mathbb{C}$ is the Fourier transform of some $H(\cdot,s) \in L^2(\mathbb{R})$,

$$\widehat{H}(k,s) = \int_{\mathbb{R}} e^{-ikx}H(x,s)dx;$$

the map $(x,s) \to H(x,s)$ being measurable on $\mathbb{R}\times[-1,1]$. Moreover, for each $\nu \in (-p_0,p_0)$, the linear map $f \to H \star f$ is well defined from $E^\nu_0(C[-1,1])$ into *itself and is uniformly bounded if ν is restricted to be in any compact subset of $(-p_0, p_0)$. Here, the convolution is taken with respect to the real variable x only,*

$$(H \star f)(x,s) = \int_{\mathbb{R}} H(x-y,s) f(y,s) dy.$$

If in addition $|k^2 \widehat{H}(k,s)|$ is bounded in $\{(k,s) \in \mathbb{C} \times [-1,1] : \operatorname{Im} k \in [-\delta, \delta]\}$ for all $\delta \in (0, p_0)$, then the map $f \to H \star f$ is well defined from $E_0^\nu(C[-1,1])$ into $E_1^\nu(C[-1,1])$ and is uniformly bounded for if ν is restricted to be in any compact subset of $(-p_0, p_0)$.

Proof. We remark that if $\widehat{H}(k,s)$ and $f(x,s)$ are both independent of s, this proposition is essentially [11, Lemma 3]. Let $0 < \delta < p_0$. We have $(1 + |k|^2)^{1/2} |\widehat{H}(k,s)| \le C$ on $\{(k,s) \in \mathbb{R} \times [-1,1] : \operatorname{Im} k \in [-\delta, \delta]\}$ and $\widehat{H}(\cdot, s) \in L^2(\mathbb{R})$. Hence, for every $s \in [-1,1]$, $\widehat{H}(\cdot, s)$ is the Fourier transform of some $H(\cdot, s) \in L^2(\mathbb{R})$, the map $(x,s) \mapsto H(x,s)$ being measurable. Moreover, by the Cauchy theorem on contour integrals in the complex plane,

$$\begin{aligned} e^{\delta x} H(x,s) &= \frac{1}{2\pi} e^{\delta x} \int_{\mathbb{R}} e^{ixk} \widehat{H}(k,s) dk \\ &= \frac{1}{2\pi} e^{\delta x} \int_{\mathbb{R}} e^{ix(i\delta + k)} \widehat{H}(i\delta + k, s) dk = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixk} \widehat{H}(i\delta + k, s) dk, \end{aligned}$$

and thus, by Plancherel,

$$||e^{\delta \cdot} H(\cdot, s)||_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} ||\widehat{H}(i\delta + \cdot, s)||_{L^2(\mathbb{R})}.$$

The same estimate with δ replaced by $-\delta$ gives

$$\sup_{s \in [-1,1]} ||\, e^{\delta |\cdot|} H(\cdot, s) \,||_{L^2(\mathbb{R})} < \infty. \tag{45}$$

Let $|\nu| < \delta$, $s \in [-1,1]$ and convolutions be only with respect to x. As in [11], we get for all $f \in E_0^\nu(C[-1,1])$

$$\begin{aligned} &\sup_{(x,s) \in \mathbb{R} \times [-1,1]} e^{-\nu|x|} \left| \int_{\mathbb{R}} H(x-y,s) f(y,s) dy \right| \\ &\quad \le ||f||_{E_0^\nu(C[-1,1])} \sup_{(x,s) \in \mathbb{R} \times [-1,1]} \int_{\mathbb{R}} e^{-\nu|x| + \nu|y| - \delta|x-y|} \left| e^{\delta|x-y|} H(x-y,s) \right| dy \\ &\le ||f||_{E_0^\nu(C[-1,1])} \sup_{s \in [-1,1]} ||\, e^{\delta|\cdot|} H(\cdot, s) \,||_{L^2(\mathbb{R})} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} e^{2\nu(|y| - |x|) - 2\delta|x-y|} dy \right)^{1/2} \\ &\le ||f||_{E_0^\nu(C[-1,1])} \sup_{s \in [-1,1]} ||\, e^{\delta|\cdot|} H(\cdot, s) \,||_{L^2(\mathbb{R})} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} e^{2|\nu|(|y-x|) - 2\delta|x-y|} dy \right)^{1/2} \\ &\qquad = ||f||_{E_0^\nu(C[-1,1])} \sup_{s \in [-1,1]} ||\, e^{\delta|\cdot|} H(\cdot, s) \,||_{L^2(\mathbb{R})} (\delta - |\nu|)^{-1/2}\,. \end{aligned}$$

If in addition $|k^2\widehat{H}(k,s)|$ is bounded in $\{(k,s)\in\mathbb{C}\times[-1,1] : \operatorname{Im} k\in[-\delta,\delta]\}$, then we can apply the previous argument to $ik\widehat{H}=\widehat{\partial_x H}$ instead of $\widehat{H}$, noting $\partial_x(H\star f)=(\partial_x H)\star f$. $\square$

Recall the dispersion function $D(k)=-c^2k^2+2(1-\cos k)+\alpha$ and let

$$p_0 := \inf\{|\operatorname{Im} k| : D(k)=0,\ \operatorname{Im} k\neq 0\}>0. \tag{46}$$

By Lemma 1 in [11], $p_0>0$.

Proposition A.2. *Let $\nu\in(-p_0,0)$. If $Q\in E_0^\nu(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} Q(x)\sin(k_0x)dx=\int_{\mathbb{R}} Q(x)\cos(k_0x)dx=0, \tag{47}$$

then, for all $c\leq 1$ close enough to 1, there exists a unique function $r\in E_1^\nu(\mathbb{R})$ such that $Lr=Q$. Moreover, the map $Q\to r$ is bounded as map $E_0^\nu(\mathbb{R})\to E_1^\nu(\mathbb{R})$ and $r\in H^2(\mathbb{R})$.

Proof. Let us formally define the function r by its Fourier representation $\widehat{r}(k):=\widehat{Q}(k)/D(k)$. As D vanishes on $\mathbb{R}$ exactly at $\pm k_0$ with non-vanishing derivative $D'(\pm k_0)=\pm(-2c^2k_0+2)\neq 0$, we can define the function

$$f(k):=\frac{-2c^2k_0+2}{2k_0}(k^2-k_0^2),$$

which also vanishes exactly at $\pm k_0$ and satisfies there $f'(\pm k_0)=D'(\pm k_0)$. Thus, we can write

$$\frac{1}{D(k)}=\frac{1}{f(k)}+\widehat{H}(k)$$

with a remainder function $\widehat{H}(k)$. Clearly $\widehat{H}(k)$ is analytic in the strip $\{k\in\mathbb{C} : \operatorname{Im} k\in(-p_0,p_0)\}$. As $|k^2/D(k)|$ is bounded in

$$\{k\in\mathbb{C} : \operatorname{Im} k\in(-p_0,p_0),\ |D(k)|>1\},$$

we know that $|k^2\widehat{H}(k)|$ is bounded on the strip $\{k\in\mathbb{C} : \operatorname{Im} k\in(-\delta,\delta)\}$ for all $\delta\in(0,p_0)$. Thus Proposition A.1 applied to the case when $\widehat{H}$ and f do not depend on the second variable s, the map $Q\mapsto H\star Q$ is well defined and bounded from $E_0^\nu(\mathbb{R})$ to $E_1^\nu(\mathbb{R})$. Moreover, $H\star Q$ is clearly in $H^2(\mathbb{R})$ as Q is assumed to decay exponentially. Note that $H\in H^1(\mathbb{R})$.

On the other hand, we ignore f for the moment and notice that the function $\frac{1}{k_0^2-k^2}\widehat{Q}(k)$ is related to the Fourier transform of the solution $r_0(x)$ of the equation $L_0r_0=r_0''+k_0^2r_0=Q$. The variation of constants formula and (47) give

$$r_0(x)=\frac{1}{k_0}\int_{-\infty}^{x}\sin(k_0(x-y))Q(y)dy=\frac{1}{k_0}\int_{x}^{\infty}\sin(k_0(y-x))Q(y)dy$$

with

$$r_0'(x) = \int_{-\infty}^{x} \cos(k_0(x-y))Q(y)dy = -\int_{x}^{\infty} \cos(k_0(y-x))Q(y)dy.$$

It easily follows that $r_0 \in E_1^\nu(\mathbb{R})$, $r_0 \in H^2(\mathbb{R})$ and that the map $Q \mapsto r_0$ is bounded as map $E_0^\nu(\mathbb{R}) \to E_1^\nu(\mathbb{R})$ and $r \in H^2(\mathbb{R})$.

Combining the two previous steps and noting that the solution r can, by definition of $\widehat{H}$, be written as $r = -\frac{2k_0}{-2c^2k_0+2} r_0 + H \star Q$, we have proved the claim. $\square$

We remark that in the previous Proposition, the hypothesis that Q is continuous is actually not used; it suffices to assume that $Q \in L^\infty(\mathbb{R})$ and $e^{|\nu \cdot|}Q \in L^\infty(\mathbb{R})$.

In the same way, one gets the following theorem, in which the assumption $Q \in E_0^\nu(\mathbb{R})$ is replaced by $e^{|\nu \cdot|}Q \in L^2(\mathbb{R})$.

Proposition A.3. *Suppose that $\nu \in (-p_0, 0)$. If $Q \in L^2(\mathbb{R})$, $e^{|\nu \cdot|}Q \in L^2(\mathbb{R})$ and*

$$\int_{\mathbb{R}} Q(x)\sin(k_0 x)dx = \int_{\mathbb{R}} Q(x)\cos(k_0 x)dx = 0,$$

then, for all $c \leq 1$ close enough to 1, there exists a unique function $r \in E_1^\nu(\mathbb{R})$ such that $Lr = Q$. Moreover, $r \in H^2(\mathbb{R})$ and

$$||r||_{E_1^\nu(\mathbb{R})} \leq C\, ||e^{|\nu \cdot|}Q||_{L^2(\mathbb{R})}.$$

Proof. With H as in the proof of Proposition A.2, let us check that

$$||H \star Q||_{E_0^\nu(\mathbb{R})} \leq C\, ||e^{|\nu \cdot|}Q||_{L^2(\mathbb{R})}$$

for all negative $-\delta < \nu < 0$. Indeed,

$$\begin{aligned}
e^{|\nu x|}|(H \star Q)(x)| &= e^{|\nu x|} \left| \int_{\mathbb{R}} H(x-y)Q(y)dy \right| \\
&\leq \left(\sup_{x,y \in \mathbb{R}} e^{|\nu x| - \delta|x-y| - |\nu y|} \right) \int_{\mathbb{R}} e^{\delta|x-y|}|H(x-y)|e^{|\nu y|}|Q(y)|dy \\
&\leq ||e^{\delta|\cdot|}H||_{L^2(\mathbb{R})}||e^{|\nu \cdot|}Q||_{L^2(\mathbb{R})},
\end{aligned}$$

where we used $|\,|\nu t| - |\nu \tau|\,| \leq \delta|t-\tau|$ (see also (45) for an H independent of s). Similarly one can prove that

$$||H' \star Q||_{E_0^\nu(\mathbb{R})} \leq ||e^{|\nu \cdot|}Q||_{L^2(\mathbb{R})}.$$

Finally, for the solution r_0 of $L_0 r_0 = Q$, the variation of constants formula implies that $r_0 \in E_1^\nu(\mathbb{R})$ and $||r_0||_{E_1^\nu(\mathbb{R})} \leq C\, ||e^{|\nu \cdot|}Q||_{L^2(\mathbb{R})}$. $\square$

We also use the following result, which is proved in [3, Proposition A.2].

Proposition A.4. *If $u_o \in C(\mathbb{R})$ satisfies* (16) *and $c > k_0^{-1/2}$, then*

$$\int_{\mathbb{R}} \sin(k_0 x)(c^2 u_o'' - \Delta_D u_o + \alpha u_o)dx = -2c^2 k_0 + 2 < 0.$$

We repeat the proof for the sake of completeness.

Proof. Two integrations by parts and the identity $L\sin(k_0 x) = 0$ give

$$\begin{aligned}
&\lim_{R\to\infty} \int_{-R}^{R} \sin(k_0 x)(c^2 u_o'' - \Delta_D u_o + \alpha u_o)dx \\
&= \lim_{R\to\infty} \int_{-R}^{R} \Big(c^2 \frac{d^2}{dx^2}\sin(k_0 x) - \Delta_D \sin(k_0 x) + \alpha \sin(k_0 x)\Big) u_o\, dx \\
&+ \lim_{x\to\infty} c^2 \{\sin(k_0 x)u_0'(x) - k_0 \cos(k_0 x) u_o(x) \\
&\qquad\qquad - \sin(-k_0 x)u_0'(-x) + k_0\cos(-k_0 x)u_o(-x)\} \\
&- \lim_{R\to\infty} \left(\int_{-R+1}^{R+1} - \int_{-R}^{R}\right) \sin(k_0(x-1))u_o(x)dx \\
&- \lim_{R\to\infty} \left(\int_{-R-1}^{R-1} - \int_{-R}^{R}\right) \sin(k_0(x+1))u_o(x)dx
\end{aligned}$$

$$\begin{aligned}
&\overset{(16)}{=} \lim_{x\to\infty} c^2\{-k_0 \sin^2(k_0 x) - k_0\cos^2(k_0 x) - k_0 \sin^2(k_0 x) - k_0 \cos^2(k_0 x)\} \\
&- \lim_{R\to\infty} \left[\int_{R}^{R+1} \sin(k_0(x-1))\cos(k_0 x)dx + \int_{-R}^{-R+1} \sin(k_0(x-1))\cos(k_0 x)dx\right] \\
&+ \lim_{R\to\infty} \left[\int_{-R-1}^{-R} \sin(k_0(x+1))\cos(k_0 x)dx + \int_{R-1}^{R} \sin(k_0(x+1))\cos(k_0 x)dx\right] \\
&= -2c^2 k_0 + \lim_{R\to\infty}\left(\int_{R-1}^{R+1} \cos^2(k_0 x)dx + \int_{-R-1}^{-R+1} \cos^2(k_0 x)dx\right) \\
&= -2c^2 k_0 + 2 < 0.
\end{aligned}$$

□

B Application of centre manifold theory

Let Y be any Banach space such that $\mathbb{D} \subset Y \subset \mathbb{H}$ with continuous embeddings (but not necessarily dense). To check the hypotheses in [28], it suffices to check that, for all $\nu \in [0, p_0)$ and all $G = (G_0, G_1, G_2) \in E_0^\nu(Q_h Y)$, there exists a unique $U = (u, v, W) \in E_0^\nu(Q_h\mathbb{D}) \cap C^1(\mathbb{R}, Q_h\mathbb{H})$ such that

$$\partial_t U = L_{\gamma,\tau} U + G. \tag{48}$$

The constant p_0 can be as in (46), or any smaller positive constant. Writing (48) as $U = KG$, we also need to check (as required in [28]) that $K \in \mathcal{L}(E_0^\nu(Q_hY), E_0^\nu(Q_h\mathbb{D}))$ and

$$||K||_\nu \leq \widetilde{\gamma}(\nu)$$

for some continuous function $\widetilde{\gamma}\colon [0, p_0) \to [0, \infty)$.

In [11], this is proved when $G(t)$ is of the particular form $G(t) = Q_h(0, G_1(t), 0)$ and this is sufficient for the proof of [28] to work. However, to fulfil the hypotheses of the statement of [28], this should be proved at least for the more general case $G(t) \in Q_h\mathbb{D}$, with the advantage that more general equations could be dealt with; see the remark after (6) in [11]. For completeness, let us check this hypothesis for all $G \in E_0^\nu(Q_h\mathbb{D})$, that is, $Y = \mathbb{D}$, following the same method as in [11]. Its validity is an obvious consequence of Theorem B.1 below.

Let us assume that $\nu \in (-p_0, p_0)$ and let $G = (G_0, G_1, G_2) \in E_0^\nu(Q_h\mathbb{D})$. The condition $G(t) \in Q_h\mathbb{D}$ is equivalent to the set of four conditions (see Lemma 2 in [11]): $G_2(t, \cdot) \in C^1[-1, 1]$, $G_0(t) = G_2(t, 0)$

$$k_0 G_0(t) = \gamma\tau^2 \int_0^1 \sin(k_0(1-s))[G_2(t,s) + G_2(t,-s)]ds \tag{49}$$

and

$$G_1(t) = \gamma\tau^2 \int_0^1 \cos(k_0(1-s))[G_2(t,s) - G_2(t,-s)]ds. \tag{50}$$

For $G_2(t) = G_2(t, \cdot) \in C^1[-1, 1]$, $G_2(t)$ is the last component of some $G(t) \in Q_h\mathbb{D}$ if and only if

$$k_0 G_2(t, 0) = \gamma\tau^2 \int_0^1 \sin(k_0(1-s))[G_2(t,s) + G_2(t,-s)]ds. \tag{51}$$

Theorem B.1. *Let the constant $p_0 > 0$ be as in* (46)*.*

1) For every $\nu \in (-p_0, p_0)$, consider the bounded linear map that sends $G_2 \in E_0^\nu(C^1[-1, 1])$ satisfying (51) *to $G = (G_0, G_1, G_2) \in E_0^\nu(Q_h\mathbb{D})$ with G_0 and G_1 given by* (49) *and* (50)*. There exists a bounded linear map*

$$\widetilde{K} : G_2 \mapsto U \in E_0^\nu(\mathbb{D})$$

defined for G_2 as above such that

$$U \in C^1(\mathbb{R}, \mathbb{H}) \quad \text{and} \quad \partial_t U = L_{\gamma,\tau} U + G.$$

2) Moreover, $U \in E_0^\nu(Q_h\mathbb{D}) \cap C^1(\mathbb{R}, Q_h\mathbb{H})$.

3) The solution U is unique in $E_0^\nu(Q_h\mathbb{D}) \cap C^1(\mathbb{R}, Q_h\mathbb{H})$.

4) We have $\widetilde{K} \in \mathcal{L}\Big(\{G_2 \in E_0^\nu(C^1[-1, 1]) : (51) \text{ holds}\},\ E_0^\nu(Q_h\mathbb{D})\Big)$ and

$$||\widetilde{K}||_\nu \leq \widetilde{\gamma}(\nu)$$

for some continuous function $\widetilde{\gamma} : [0, p_0) \to [0, \infty)$.

We shall prove this theorem at the end of this appendix. First we state a lemma, the proof of which is elementary and hence omitted.

Lemma B.2. *Let $G_2 \in E_0^\nu(C[-1,1])$ with $\nu \in (-p_0, p_0)$. For each $s \in [-1,1]$, let the function $\widetilde{G}_2(\cdot, s)$ be defined as follows. Let $\kappa \in C_0^\infty(\mathbb{R}, [0,\infty))$ be such that $\int_{\mathbb{R}} \kappa(t)dt = 1$ and set*

$$\begin{aligned}\widetilde{G}_2(t,s) = \cosh(p_0 t)\Big(\int_{-\infty}^{t} & G_2(u,s)/\cosh(p_0 u)\,du \\ & - \int_{-\infty}^{t} \kappa(u)du \int_{\mathbb{R}} G_2(u,s)/\cosh(p_0 u)\,du \Big) \\ = \cosh(p_0 t)\Big(- & \int_{t}^{\infty} G_2(u,s)/\cosh(p_0 u)\,du \\ & + \int_{t}^{\infty} \kappa(u)du \int_{\mathbb{R}} G_2(u,s)/\cosh(p_0 u)\,du \Big).\end{aligned}$$

Then

1) $\widetilde{G}_2 \in E_1^\nu(C[-1,1])$,

2) $\partial_t \widetilde{G}_2(t,s) = G_2(t,s)$
$$- \cosh(p_0 t)\kappa(t) \int_{\mathbb{R}} G_2(u,s)/\cosh(p_0 u)\,du + p_0 \tanh(p_0 t)\widetilde{G}_2(t,s),$$

3) $G_2 - \partial_t \widetilde{G}_2 \in E_1^\nu(C[-1,1])$,

4) $\widetilde{\widetilde{G}}_2(t) := \partial_t \Big(G_2(t) - \partial_t \widetilde{G}_2(t,s) \Big)$
$$= \{p_0 \sinh(p_0 t)\kappa(t) + \cosh(p_0 t)\kappa'(t)\} \int_{\mathbb{R}} G_2(u,s)/\cosh(p_0 u)\,ds$$
$$- (p_0/\cosh(p_0 t))^2 \widetilde{G}_2(t,s) - p_0 \tanh(p_0 t)\partial_t \widetilde{G}_2(t,s) \in E_0^\nu(C[-1,1]),$$

5) *the linear maps $G_2 \ni E_0^\nu(C[-1,1]) \to \widetilde{G}_2 \in E_1^\nu(C[-1,1])$ and $G_2 \ni E_0^\nu(C[-1,1]) \to \widetilde{\widetilde{G}}_2 \in E_0^\nu(C[-1,1])$ are bounded.*

Let us consider the last component of equation (48).

Proposition B.3. *Given $G \in E_0^\nu(Q_h \mathbb{D})$, let $U = (u, v, W) \in E_0^\nu(\mathbb{D}) \cap C^1(\mathbb{R}, \mathbb{H})$ be a solution to (48). Then $u \in E_1^\nu(\mathbb{R})$ and W solves the equation*

$$\partial_t W = \partial_s W + G_2, \quad W(t,0) = u(t),$$

which has the unique solution $W \in E_0^\nu(C^1[-1,1]) \cap C^1(\mathbb{R}, C[-1,1])$ given by

$$W(t,s) = u(t+s) - \int_t^{t+s} G_2(\sigma, t+s-\sigma)d\sigma.$$

Moreover, this defines an affine map $G_2 \to W$ such that

$$||W||_{E_0^\nu(C^1[-1,1])} \leq C\,\left(||u||_{E_1^\nu(\mathbb{R})} + ||G_2||_{E_0^\nu(C^1[-1,1])} \right). \tag{52}$$

Proof. Clearly, the given function W is a solution and the estimate holds for this W. To check uniqueness, it is enough to consider the case $u = 0$ and $G_2 = 0$. If W is a solution, let $\widetilde{W}(t,s) = W(t-s,s)$, that is, $W(t,s) = \widetilde{W}(t+s,s)$. Then W and $\widetilde{W}$ are $C^1(\mathbb{R}\times[-1,1])$, and $\partial_s\widetilde{W}(t,s) = 0$ with $\widetilde{W}(t,0) = 0$. Hence $\widetilde{W} = 0$. $\square$

Thanks to Proposition B.3, (48) becomes

$$\begin{aligned}
\partial_t u &= v + G_0,\\
\partial_t v &= \gamma\tau^2\Delta_D u - \tau^2 u\\
&\qquad - \gamma\tau^2\int_t^{t+1} G_2(s,t+1-s)ds - \gamma\tau^2\int_t^{t-1} G_2(s,t-1-s)ds + G_1\\
&= \gamma\tau^2\Delta_D u - \tau^2 u - \gamma\tau^2\int_0^1 G_2(t+1-s,s)ds\\
&\qquad - \gamma\tau^2\int_0^{-1} G_2(t-1-s,s)ds + G_1.
\end{aligned}$$

Thus, we need to find $u \in E_1^\nu(\mathbb{R})$ such that $\partial_t u - G_0 \in C^1(\mathbb{R})$ and solving

$$\begin{aligned}
\partial_t(\partial_t u - G_0) = \gamma\tau^2\Delta_D u - \tau^2 u - \gamma\tau^2\int_0^1 G_2(t+1-s,s)ds&\\
- \gamma\tau^2\int_0^{-1} G_2(t-1-s,s)ds + G_1\,.&
\end{aligned}\tag{53}$$

If in addition $G_2 \in E_1^\nu(C[-1,1])$, (49) implies $G_0 \in E_1^\nu(\mathbb{R})$ and the equation reads (for $u \in E_1^\nu(\mathbb{R})\cap C^2(\mathbb{R})$ now)

$$\begin{aligned}
Lu &:= \gamma^{-1}\tau^{-2}u'' - \Delta_D u + \gamma^{-1}u\\
&= -\int_0^1 G_2(t+1-s,s)ds - \int_0^{-1} G_2(t-1-s,s)ds\\
&\qquad + \gamma^{-1}\tau^{-2}G_1 + \gamma^{-1}\tau^{-2}G_0'\\
&\overset{(49),(50)}{=} -\int_0^1 G_2(t+1-s,s)ds + \int_0^1 G_2(t-1+s,-s)ds\\
&\qquad + \int_0^1 \cos(k_0(1-s))[G_2(t,s) - G_2(t,-s)]ds\\
&\qquad + k_0^{-1}\int_0^1 \sin(k_0(1-s))[\partial_t G_2(t,s) + \partial_t G_2(t,-s)]ds =: Q(G_2).
\end{aligned}\tag{54}$$

Proposition B.4. *If $\nu \in (-p_0,0)$ and $G_2 \in E_1^\nu(C[-1,1])$, then equation (54) has a solution $u \in E_1^\nu(\mathbb{R})\cap C^2(\mathbb{R})$ such that*

$$||u||_{E_1^\nu(\mathbb{R})} \le C\,||G_2||_{E_0^\nu(C[-1,1])}$$

uniformly in ν on compact subsets of $(-p_0,0)$.

Remark. We require $G_2 \in E_1^\nu(C[-1,1])$ in the hypotheses and thus (54) makes sense. However, in the conclusion, the weaker norm $||\cdot||_{E_0^\nu(C[-1,1])}$ is used. As the norm $||G_2||_{E_0^\nu(C^1[-1,1])}$ is needed in (52) to control $||W||_{E_0^\nu(C^1[-1,1])}$, in the end the norm in the statement of Theorem B.1 is $||G||_{E_0^\nu(C^1[-1,1])}$.

Proof. As $\nu \in (-p_0, 0)$, we can consider the Fourier transform $\widehat{G}_2(k,s)$ of $G_2(t,s)$ with respect to t. The Fourier transform $\mathcal{F}[Q(G_2)]$ of the right-hand side of (54) is

$$\begin{aligned}
&\int_0^1 \Big(- e^{ik(1-s)} + \cos(k_0(1-s)) + k_0^{-1}\sin(k_0(1-s))ik\Big)\widehat{G}_2(k,s)ds \\
&\qquad + \int_0^1 \Big(e^{ik(-1+s)} - \cos(k_0(1-s)) + k_0^{-1}\sin(k_0(1-s))ik\Big)\widehat{G}_2(k,-s)ds \\
&= \int_0^1 \Big(\{\cos(k_0(1-s)) - \cos(k(1-s))\} \\
&\qquad\qquad + i\{k_0^{-1}\sin(k_0(1-s))k - \sin(k(1-s))\}\Big)\widehat{G}_2(k,s)ds \\
&\qquad + \int_{-1}^0 \Big(\{\cos(k(1+s)) - \cos(k_0(1+s))\} \\
&\qquad\qquad + i\{k_0^{-1}\sin(k_0(1+s))k - \sin(k(1+s))\}\Big)\widehat{G}_2(k,s)ds \\
&= \int_{-1}^1 \Big(\mathrm{sgn}(s)\{\cos(k_0(1-|s|)) - \cos(k(1-|s|))\}\widehat{G}_2(k,s) \\
&\qquad\qquad + \{\mathrm{sinc}(k_0(1-|s|)) - \mathrm{sinc}(k(1-|s|))\}(1-|s|)ik\widehat{G}_2(k,s)\Big)ds,
\end{aligned}$$

where $\mathrm{sgn}(0) = 0$ and sinc is the cardinal sine function, i.e., $\mathrm{sinc}(k) = \sin(k)/k$ ($= 1$ at $k = 0$).

Let $\widetilde{G}_2(\cdot, s)$ and $\widetilde{\widetilde{G}}_2$ be as in Lemma B.2. As $G_2 \in E_1^\nu(C[-1,1])$, $\widetilde{G}_2 \in E_2^\nu(C[-1,1])$. Because of $\partial_t G_2 = \partial_t^2 \widetilde{G}_2 + \widetilde{\widetilde{G}}_2$, we have for $\nu \in (-p_0, 0)$

$$ik\widehat{G}_2 = -k^2\mathcal{F}[\widetilde{G}_2] + \mathcal{F}[\widetilde{\widetilde{G}}_2]\,,$$

where the Fourier transforms are taken with respect to the first variable only. Hence

$$\begin{aligned}
\mathcal{F}[Q(G_2)] = &\int_{-1}^1 \Big(\mathrm{sgn}(s)\{\cos(k_0(1-|s|)) - \cos(k(1-|s|))\}\widehat{G}_2(k,s) + \\
&\{\mathrm{sinc}(k_0(1-|s|)) - \mathrm{sinc}(k(1-|s|))\}(1-|s|)\{-k^2\mathcal{F}[\widetilde{G}_2](k,s) + \mathcal{F}[\widetilde{\widetilde{G}}_2](k,s)\}\Big)ds.
\end{aligned}$$

Define

$$\widehat{G}_3(k) := \int_{-1}^1 \{\mathrm{sinc}(k_0(1-|s|)) - \mathrm{sinc}(k(1-|s|))\}(1-|s|)\mathcal{F}[\widetilde{G}_2](k,s)ds. \quad (55)$$

Clearly, $\widehat{G}_3(\pm k_0) = 0$. At the end of the proof, we shall check that $G_3 \in E_2^\nu(\mathbb{R})$.

To analyse the left-hand side of (54), we consider

$$L(u - \gamma\tau^2 G_3) = \gamma^{-1}\tau^{-2}(u - \gamma\tau^2 G_3)'' - \Delta_D(u - \gamma\tau^2 G_3) + \gamma^{-1}(u - \gamma\tau^2 G_3),$$

whose Fourier transform, using (55), equals to

$$\begin{aligned}\mathcal{F}[L(u - \gamma\tau^2 G_3)] = \mathcal{F}[Q(G_2)] - \gamma\tau^2 D(k)\widehat{G}_3(k) = \gamma\tau^2(2\cos(k) - 2 - \gamma^{-1})\widehat{G}_3 \\ + \int_{-1}^{1} \Big(\operatorname{sgn}(s)\{\cos(k_0(1-|s|)) - \cos(k(1-|s|))\}\widehat{G}_2(k,s) \\ + \{\operatorname{sinc}(k_0(1-|s|)) - \operatorname{sinc}(k(1-|s|))\}(1-|s|)\mathcal{F}(\widetilde{\widetilde{G}}_2)(k,s)\}\Big) ds.\end{aligned}$$

Note that by construction, the Fourier transform above vanishes at $k = \pm k_0$, the only real roots of the dispersion function D. Hence, by (55),

$$\begin{aligned}\mathcal{F}[u - \gamma\tau^2 G_3](k) = \int_{-1}^{1} \widehat{H}_1(k,s)\widehat{G}_2(k,s) ds + \int_{-1}^{1} \widehat{H}_2(k,s)\mathcal{F}(\widetilde{\widetilde{G}}_2)(k,s) ds \\ + \int_{-1}^{1} \widehat{H}_3(k,s)\mathcal{F}(\widetilde{G}_2)(k,s) ds, \quad (56)\end{aligned}$$

where $\widehat{H}_j(k,s)$ is continuous in (k,s) for $s \neq 0$, the function $k \to \widehat{H}_j(k,s)$ is analytic in the strip $\{k \in \mathbb{C} : \operatorname{Im} k \in (-p_0, p_0)\}$ and $(1+|k|^2)H_j(k,s)$ is bounded in $\{(k,s) \in \mathbb{C} \times [-1,1] : \operatorname{Im} k \in [-\delta,\delta]\}$ for $j = 1,2,3$. For example,

$$\widehat{H}_1(k,s) = \frac{\operatorname{sgn}(s)\{\cos(k_0(1-|s|)) - \cos(k(1-|s|))\}}{D(k)}.$$

Again by Proposition A.1, the map that sends $G_2 \mapsto (G_2, \widetilde{\widetilde{G}}_2, \widetilde{G}_2) \mapsto u - \gamma\tau^2 G_3 \in E_1^\nu(\mathbb{R}) \cap H^2(\mathbb{R})$ is well defined and bounded from $\Big(E_0^\nu(C[-1,1])\Big)^3$ to $E_1^\nu(\mathbb{R})$,

$$\begin{aligned}u - \gamma\tau^2 G_3 = \int_{-1}^{1} H_1(\cdot,s) \star G_2(\cdot,s) ds + \int_{-1}^{1} H_2(\cdot,s) \star \widetilde{\widetilde{G}}_2(\cdot,s) ds \\ + \int_{-1}^{1} H_3(\cdot,s) \star \widetilde{G}_2(\cdot,s) ds, \quad (57)\end{aligned}$$

where the convolutions are with respect to the first variable only. Moreover u is a solution to (54). By Proposition A.1,

$$||u - \gamma\tau^2 G_3||_{E_1^\nu(\mathbb{R})} \leq C \left(||G_2||_{E_0^\nu(C[-1,1])} + ||\widetilde{\widetilde{G}}_2||_{E_0^\nu(C[-1,1])} + ||\widetilde{G}_2||_{E_0^\nu(C[-1,1])} \right)$$

and

$$||G_3||_{E_1^\nu(\mathbb{R})} \leq C_1 ||\widetilde{G}_2||_{E_1^\nu(C[-1,1])} \leq C_2 ||G_2||_{E_0^\nu(C[-1,1])}$$

uniformly in ν in compact subsets of $(-p_0, 0)$. To see that C_1 is finite, rewrite (55) once more

$$\widehat{G}_3(k) = \int_{-1}^{1} \operatorname{sinc}(k_0(1-|s|))(1-|s|)\mathcal{F}[\widetilde{G}_2](k,s)ds \\ - \int_{-1}^{1} \operatorname{sinc}(k(1-|s|))(1-|s|)\mathcal{F}[\widetilde{G}_2](k,s)ds,$$

and observe that $|\operatorname{sinc}(k(1-|s|))(1-|s|)| \leq C\,|k|^{-1}$ on $\{(k,s) \in \mathbb{C} \times [-1,1] : \operatorname{Im} k \in (-p_0, p_0)\}$. Then Proposition A.1 allows to transform back

$$G_3 = \int_{-1}^{1} \operatorname{sinc}(k_0(1-|s|))(1-|s|)\widetilde{G}_2(\cdot,s)ds \\ - \int_{-1}^{1} \mathcal{F}^{-1}\{\operatorname{sinc}(\cdot(1-|s|))(1-|s|)\} \star \widetilde{G}_2(\cdot,s)ds, \quad (58)$$

with $||G_3||_{E_1^\nu(\mathbb{R})} \leq C_1 ||\widetilde{G}_2||_{E_1^\nu(C[-1,1])}$ for a finite constant C_1. Moreover $G_3 \in E_2^\nu(\mathbb{R})$ as $\widetilde{G}_2 \in E_2^\nu(C[-1,1])$. □

Proof of Theorem B.1

Proposition B.4 ensures the existence of a solution $u \in E_1^\nu(\mathbb{R})$ to equations (49), (50), (53) for $G_2 \in E_1^\nu(C[-1,1])$ and $\nu \in (-p_0, 0)$. However these equations also make sense for $G_2 \in E_0^\nu(C[-1,1])$. By an approximation procedure, the existence of a solution u to (49), (50), (53) and the estimate

$$||u||_{E_1^\nu(\mathbb{R})} \leq C\,||G_2||_{E_0^\nu(C[-1,1])}. \quad (59)$$

of Proposition B.4 remain true for all $G_2 \in E_0^\nu(C[-1,1])$ (uniformly in ν in compact subsets of $(-p_0, 0)$) The approximation procedure thus defines a bounded linear map $G_2 \mapsto u$.

This linear map is well-defined also when $\nu \in [0, p_0)$, the constants being in fact uniform in ν in every compact subset of $(-p_0, p_0)$ (see (57), (58), Proposition A.1 and Lemma B.2) but it must be checked that u also gives rise to a solution when $\nu \in [0, p_0)$. This can be done by a truncation that brings the case $\nu \in [0, p_0)$ back to the case $\nu \in (-p_0, 0)$. Namely let $\nu \in [0, p_0)$, $\nu_+ = (\nu + p_0)/2$ and $\nu_- = (-p_0 - \nu)/2$. Let $\zeta \in C_0^\infty(\mathbb{R}, [0,\infty))$ be equal to 1 in a neighbourhood of 0. Then, for $G_2 \in E_0^\nu(C[-1,1])$, the sequence $\{\zeta(\cdot/n)G_2\}_{n\geq 1} \subset E_0^{\nu_-}(C[-1,1])$ converges to G_2 in $E_0^{\nu_+}(C[-1,1])$ and is bounded in $E_0^\nu(C[-1,1])$. Hence it is a Cauchy sequence in $E_0^{\nu_+}(C[-1,1])$. The corresponding sequence $\{u_n\}_{n\geq 1} \subset E_1^{\nu_-}(\mathbb{R})$ therefore converges in $E_1^{\nu_+}(\mathbb{R})$ to some $u \in E_1^\nu(\mathbb{R})$. As each u_n solves (49), (50), (53) with the right-hand sides defined from $\zeta(\cdot/n)G_2$, it follows that u solves (49), (50), (53) with the right-hand sides defined from G_2, giving rise in this way to a bounded linear map $G_2 \mapsto u$. This proves the first part of Theorem B.1.

Let us prove the second part of Theorem B.1. Firstly, assume that $\nu \in (-p_0, 0)$. If $G \in E_0^\nu(Q_h\mathbb{D})$, then (48) gives $\partial_t P_1 U = L_{\gamma,\tau} P_1 U$. As a consequence $U(t) \notin Q_h\mathbb{D}$ for some $t \in \mathbb{R}$ would imply that $P_1 U$ is a non-trivial periodic solution on the centre manifold, in contradiction with $\lim_{|t|\to\infty} ||U(t)||_{\mathbb{D}} = 0$ (as $\nu \in (-p_0, 0)$). See the paragraph before Proposition 3.3 for the fact that the centre manifold (here for the linear problem) is filled by the equilibrium and periodic solutions. The above truncation procedure allows one to conclude that $U \in E_0^\nu(Q_h\mathbb{D})$ also when $\nu \in [0, p_0)$.

Finally, we turn to the third part of Theorem B.1 about uniqueness. Let us first study the special case $G = 0$ for $\nu \in (-p_0, p_0)$. As W is uniquely determined by u (see Proposition B.3), let us consider any solution u in $E_1^\nu(\mathbb{R}) \cap C^2(\mathbb{R})$ to

$$Lu = \gamma^{-1}\tau^{-2}u'' - \Delta_D u + \gamma^{-1}u = 0.$$

Observe that $u'' \in E_0^\nu(\mathbb{R})$ and let $\nu_- \in (-p_0, -|\nu|)$. For all test functions $\tilde{u} \in H^2(\mathbb{R}) \cap E_1^{\nu_-}(\mathbb{R})$, integrations by parts gives

$$0 = \int_{\mathbb{R}} Lu \cdot \tilde{u} dt = \int_{\mathbb{R}} uL\tilde{u} dt.$$

By Proposition A.2, the map $\tilde{u} \mapsto L\tilde{u} =: Q$ is surjective from $H^2(\mathbb{R}) \cap E_1^{\nu_-}(\mathbb{R})$ to

$$\left\{ Q \in E_0^{\nu_-}(\mathbb{R}) : \int_{\mathbb{R}} Q(t)\cos(k_0 t) dt = \int_{\mathbb{R}} Q(t)\sin(k_0 t) dt = 0 \right\}.$$

Thus, for a solution u we get $\int_{\mathbb{R}} u(t)Q(t)dt = 0$ for all such Q.

Let $\eta_c, \eta_s \in E_0^{\nu_-}(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} \eta_c(t)\cos(k_0 t) dt = \int_{\mathbb{R}} \eta_s(t)\sin(k_0 t) dt = 1,$$

$$\int_{\mathbb{R}} \eta_c(t)\sin(k_0 t) dt = \int_{\mathbb{R}} \eta_s(t)\cos(k_0 t) dt = 0.$$

If $Q \in E_0^{\nu_-}(\mathbb{R})$, then

$$\widetilde{Q} := Q - \eta_c \int_{\mathbb{R}} Q(y)\cos(k_0 y) dy - \eta_s \int_{\mathbb{R}} Q(y)\sin(k_0 y) dy$$

satisfies $\int_{\mathbb{R}} \widetilde{Q}(t)\cos(k_0 t) dt = \int_{\mathbb{R}} \widetilde{Q}(t)\sin(k_0 t) dt = 0$ and therefore by Fubini

$$\begin{aligned} 0 &= \int_{\mathbb{R}} u(t) \left(Q(t) - \eta_c(t) \int_{\mathbb{R}} Q(y)\cos(k_0 y) dy - \eta_s(t) \int_{\mathbb{R}} Q(y)\sin(k_0 y) dy \right) dt \\ &= \int_{\mathbb{R}} \left(u(t) - \cos(k_0 t) \int_{\mathbb{R}} u(y)\eta_c(y) dy - \sin(k_0 t) \int_{\mathbb{R}} u(y)\eta_s(y) dy \right) Q(t) dt. \end{aligned}$$

Hence the function u is in the span of the two functions $\cos(k_0 \cdot)$ and $\sin(k_0 \cdot)$:

$$\begin{aligned} u &= \cos(k_0 \cdot) \int_{\mathbb{R}} u(y)\eta_c(y) dy + \sin(k_0 \cdot) \int_{\mathbb{R}} u(y)\eta_s(y) dy \\ &\in \operatorname{span}\{\cos(k_0 \cdot), \sin(k_0 \cdot)\} \end{aligned}$$

and, for all $t \in \mathbb{R}$, we have (see Lemma 2 in [11])

$$(u(t), u'(t), u(t+\cdot)) \in \operatorname{span}\Big\{ \Big(\cos(k_0 t), -k_0 \sin(t), \cos(k_0(t+\cdot)) \Big), \\ \Big(\sin(k_0 t), k_0 \cos(t), \sin(k_0(t+\cdot)) \Big) \Big\} = P_1 \mathbb{D},$$

where $u(t+\cdot)$ denotes the function $s \to u(t+s)$ for $s \in [-1,1]$.

We are now ready to check the uniqueness of the solution $U \in E_0^\nu(Q_h\mathbb{D}) \cap C^1(\mathbb{R}, Q_h\mathbb{H})$ for $\nu \in (-p_0, p_0)$. It is clearly sufficient to check it for $G = 0$ only. Moreover, as W is unique for a unique $u \in E_1^\nu(\mathbb{R})$ (see Prop. B.3), it is enough to show that $u = 0$ is the unique solution in $E_1^\nu(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation

$$Lu = 0 \;\text{ such that }\; \forall t \in \mathbb{R} \;\; U(t,\cdot) := (u(t), u'(t), u(t+\cdot)) \in Q_h\mathbb{D},$$

where $U(t,\cdot)$ denotes the function $s \to U(t,s)$ for $s \in [-1,1]$. As $Lu = 0$, we already know that necessarily $U(t) \in P_1\mathbb{D}$ for all $t \in \mathbb{R}$. Hence $u = 0$ as desired.

The last two claims of Theorem B.1 result from (52) and (59), where the various constants are uniform in ν on any compact subset of $(-p_0, p_0)$.

Remark. A look into the proofs of the present appendix shows that the arguments work as well for

$$Y = \{G = (G_0, G_1, G_2) : G_0, G_1 \in E_0^\nu(\mathbb{R}),\, G_2 \in E_0^\nu(C^1[-1,1]),\, (49) \text{ and } (50) \text{ hold}\}.$$

Acknowledgement JZ gratefully acknowledges funding by the EPSRC through project EP/K027743/1, the Leverhulme Trust (RPG-2013-261) and a Royal Society Wolfson Research Merit Award.

References

[1] W. Atkinson and N. Cabrera. Motion of a Frenkel-Kontorowa dislocation in a one-dimensional crystal. *Phys. Rev.*, 138(3A):A763–A766, May 1965.

[2] Oleg M. Braun and Yuri S. Kivshar. Nonlinear dynamics of the Frenkel-Kontorova model. *Phys. Rep.*, 306(1-2):108, 1998.

[3] Boris Buffoni, Hartmut Schwetlick, and Johannes Zimmer. Travelling waves for a Frenkel-Kontorova chain. `http://arxiv.org/abs/1507.03189`, 2015.

[4] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[5] Y. Y. Earmme and J. H. Weiner. Dislocation dynamics in the modified Frenkel-Kontorova model. *J. Appl. Phys.*, 48(8):3317–3331, 1977.

[6] Michal Fesckan and Vassilis M. Rothos. Travelling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions. *Nonlinearity*, 20(2):319–341, 2007.

[7] N. Flytzanis, S. Crowley, and V. Celli. High velocity dislocation motion and interatomic force law. *J. Phys. Chem. Solids*, 38(5):539–552, 1977.

[8] F. C. Frank and J. H. van der Merwe. One-dimensional dislocations. IV. dynamics. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 201(1065):261–268, 1950.

[9] J. Frenkel and T. Kontorova. On the theory of plastic deformation and twinning. *Acad. Sci. U.S.S.R. J. Phys.*, 1:137–149, 1939.

[10] Gero Friesecke and Jonathan A. D. Wattis. Existence theorem for solitary waves on lattices. *Comm. Math. Phys.*, 161(2):391–418, 1994.

[11] Gérard Iooss and Klaus Kirchgässner. Travelling waves in a chain of coupled nonlinear oscillators. *Comm. Math. Phys.*, 211(2):439–464, 2000.

[12] G. James. Centre manifold reduction for quasilinear discrete systems. *J. Nonlinear Sci.*, 13(1):27–63, 2003.

[13] Carl-Friedrich Kreiner and Johannes Zimmer. Existence of subsonic heteroclinic waves for the Frenkel-Kontorova model with piecewise quadratic on-site potential. *Nonlinearity*, 24(4):1137–1163, 2011.

[14] O. Kresse and L. Truskinovsky. Mobility of lattice defects: discrete and continuum approaches. *J. Mech. Phys. Solids*, 51(7):1305–1332, 2003.

[15] O. Kresse and L. Truskinovsky. Lattice friction for crystalline defects: from dislocations to cracks. *J. Mech. Phys. Solids*, 52(11):2521–2543, 2004.

[16] R. S. MacKay and S. Aubry. Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators. *Nonlinearity*, 7(6):1623–1643, 1994.

[17] Wen-Xin Qin. Uniform sliding states in the undamped Frenkel-Kontorova model. *J. Differential Equations*, 249(7):1764–1776, 2010.

[18] Wen-Xin Qin. Existence and modulation of uniform sliding states in driven and overdamped particle chains. *Comm. Math. Phys.*, 311(2):513–538, 2012.

[19] A. V. Savin, Y. Zolotaryuk, and J. C. Eilbeck. Moving kinks and nanopterons in the nonlinear Klein-Gordon lattice. *Phys. D*, 138(3-4):267–281, 2000.

[20] Erwin Schrödinger. Zur Dynamik elastisch gekoppelter Punktsysteme. *Ann. Phys.*, 44:916–934, 1914.

[21] Hartmut Schwetlick and Johannes Zimmer. Existence of dynamic phase transitions in a one-dimensional lattice model with piecewise quadratic interaction potential. *SIAM J. Math. Anal.*, 41(3):1231–1271, 2009.

[22] L. I. Slepyan. Solutions for nonlinear lattices. International Conference on Fracture ICF11, Italy 2005, 2005.

[23] Leonid Slepyan, Andrej Cherkaev, and Elena Cherkaev. Transition waves in bistable structures. II. Analytical solution: wave speed and energy dissipation. *J. Mech. Phys. Solids*, 53(2):407–436, 2005.

[24] D. J. Srolovitz and P. S. Lomdahl. Dislocation dynamics in the 2-D Frenkel-Kontorova model. *Phys. D*, 23(1–3):402 – 412, 1986.

[25] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[26] Lev Truskinovsky and Anna Vainchtein. Kinetics of martensitic phase transitions: lattice model. *SIAM J. Appl. Math.*, 66(2):533–553 (electronic), 2005.

[27] Lev Truskinovsky and Anna Vainchtein. Solitary waves in a nonintegrable Fermi-Pasta-Ulam chain. *Phys. Rev. E*, 90:042903, Oct 2014.

[28] A. Vanderbauwhede and G. Iooss. Center manifold theory in infinite dimensions. In *Dynamics reported: expositions in dynamical systems*, volume 1 of *Dynam. Report. Expositions Dynam. Systems (N.S.)*, pages 125–163. Springer, Berlin, 1992.